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Doctorat de l'Université de Toulouse

Analyse d'un modèle d'interaction fluide-structure : caractère bien posé, stabilisation et simulations numériques

Thèse présentée et soutenue, le 27 novembre 2025 par

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École doctorale
EDMITT - Ecole Doctorale Mathématiques, Informatique et Télécommunications de Toulouse

Spécialité
Mathématiques et Applications

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Université de Toulouse

Doctoral School **MITT: Domaine Mathématiques: Mathématiques appliquées**

University Department **Institut de Mathématiques de Toulouse (UMR5219)**

Thesis defended by **Diego Gajardo**

Defended on **November 27, 2025**

In order to become Doctor from Université de Toulouse

Academic Field **Mathematics**

Speciality **Partial differential equations**

Analysis of a fluid-structure interaction model: well-posedness, stabilization and numerical simulations

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Remerciements

Je commence par remercier Franck et Michel. Tout d'abord, pour m'avoir donné l'opportunité de travailler avec vous. Cela a été une expérience très enrichissante pour moi. Je vous remercie infiniment tous les deux pour votre gentillesse et votre pédagogie pour répondre à mes questions tout au long de cette période. De la même manière, je remercie beaucoup Jean-Pierre de nous avoir aidés à clarifier de nombreuses questions qui ont surgi tout au long de ce travail. Sa patience pour répondre à mes questions, toujours de façon pédagogique, est quelque chose que je lui suis infiniment reconnaissant. Je me sens privilégié d'avoir pu travailler en collaboration avec vous trois.

Je remercie beaucoup Muriel Boulakia et Miguel Fernández d'avoir accepté de relire et rapporter cette thèse. Je tiens également à remercier sincèrement Majdi Azaiez, Matthieu Hillairet, Jean-Pierre Raymond et Takéo Takahashi d'avoir accepté de faire partie du jury.

Mes remerciements vont également à Alberto Mercado pour m'avoir introduit dans le domaine de la recherche et pour m'avoir motivé à poursuivre un doctorat.

Je m'addresse maintenant à mes camarades. Tout d'abord, je remercie tous ceux que j'ai eu la chance de rencontrer lorsque j'étais encore étudiant à Valparaíso. De la même manière, je remercie mes camarades de thèse ici à Toulouse, avec lesquels j'ai eu le privilège de partager cette période.

Enfin, je remercie mes parents et mes frères pour leur soutien infini, que j'ai toujours ressenti malgré la distance.

Table des matières

Appendices i

Résumé i

Remerciements ii

Table des matières iii

1 Introduction 1
- 1.1 Contexte général . . . 1
- 1.2 Description du modèle étudié . . . 2
- 1.3 Résultats de la thèse . . . 5
  - 1.3.1 Chapitre 2 : Étude du système de Stokes stationnaire avec conditions aux limites mixtes dans des domaines polygonaux curvilignes non convexes . . 5
  - 1.3.2 Chapitre 3 : Existence d'une solution forte locale en temps pour le système d'interaction fluide-structure . . . 10
  - 1.3.3 Chapitre 4 : Simulations numériques du système d'interaction fluide-structure . . . 16
  - 1.3.4 Chapitre 5 : Stabilisation du système d'interaction fluide-structure . . . 22
  - 1.3.5 Chapitre 6 : Simulations numériques du problème de stabilisation du système d'interaction fluide-structure . . . 30
- 1.4 Perspectives . . . 36

2 Study of the stationary Stokes system with mixed boundary conditions in non-convex curvilinear polygonal domains **38**
- 2.1 Introduction . . . 39
  - 2.1.1 Statement of the problem . . . 39
  - 2.1.2 Motivation . . . 40
  - 2.1.3 Outline . . . 41
- 2.2 Classes of domains and functional setting . . . 41
  - 2.2.1 Classes of domains . . . 41
  - 2.2.2 Functional setting . . . 42
- 2.3 Existence, uniqueness and regularity . . . 45
- 2.4 Rewriting the Stokes system as an operator equation . . . 47
  - 2.4.1 The Leray projector . . . 47
  - 2.4.2 Stokes operator and the analyticity of its underlying semigroup . . . 51
  - 2.4.3 Expression of the pressure . . . 58
  - 2.4.4 System reformulated as an operator equation . . . 61
- Appendix A : Study of the Stokes resolvent system . . . 62
- Appendix B : Technical result adapted from Pazy's book . . . 64
- Appendix C : Justification of the identity (2.4.55) (formal computations) . . . 65

3 Existence of a local-in-time strong solution of the fluid-structure system 67
3.1 Introduction . . . 69
3.1.1 Description of the model . . . 69
3.1.2 Discussion . . . 71
3.1.3 Outline . . . 72
3.2 Notation and statement of the main result . . . 72
3.2.1 Notation . . . 73
3.2.2 System in the reference configuration . . . 74
3.2.3 Statement of the main result . . . 77
3.3 System rewritten using the semigroup formulation . . . 79
3.3.1 Steady Stokes system . . . 79
3.3.2 Euler-Bernoulli beam equation . . . 82
3.3.3 Coupled linear fluid-structure system . . . 83
3.3.4 Analyticity of the semigroup generated by $(\mathcal{A}, \mathcal{D}(A))$ on $\mathbf{Z}$ . . . 85
3.4 Existence and regularity results for the non-homogeneous linear system . . . 88
3.5 Estimates of nonlinear terms . . . 96
3.5.1 Auxiliary results . . . 96
3.5.2 Estimates of nonlinear terms . . . 100
3.6 Proof of Theorem 3.2.1 . . . 105
Appendix A: Expression of the function H and simplification in the modelling . . . 107
Discussion about the term J_3 . . . 109
Appendix B: Change of variables . . . 111
B.1 General change of variable formula . . . 111
B.2 Nonlinear terms coming from the geometric transformation (3.2.9) . . . 111
Appendix C: Auxiliar results . . . 112
C.1 Study of Stokes system with irregular divergence data (case 1) . . . 112
C.2 Study of the Stokes system with irregular divergence data (case 2) . . . 113
C.3 Auxiliar Lemma . . . 113
Appendix D: Proof of estimates (3.5.17) and (3.5.18) . . . 114
D.1 Proof of estimate (3.5.17) . . . 115
D.2 Proof of estimate (3.5.18) . . . 115

4 Numerical simulations of the fluid-structure interaction system 118
4.1 Introduction . . . 119
4.2 A variational formulation of the continuous system . . . 120
4.3 ALE mapping and the time-marching process . . . 122
4.3.1 The time-marching process . . . 123
4.3.2 Full discretization of the nonlinear system . . . 124
4.4 Numerical simulations . . . 125
4.4.1 Computation of the spectrum . . . 127
4.4.2 Solving the direct problem . . . 135
Appendix A: Terms to be linearized . . . 145

5 Stabilization of the Fluid-Structure Interaction system 147
5.1 Introduction . . . 149
5.2 Notation and statement of the main results . . . 152
5.2.1 Notation . . . 152
5.2.2 System in the reference configuration and statement of the main results . . . 154
5.3 Analysis of the fluid-structure operator . . . 158
5.3.1 Fluid operator . . . 158
5.3.2 Structure operator . . . 164
5.3.3 Fluid-Structure operator . . . 165

5.4 Characterization of the adjoint operator $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ . . . 166
5.5 Resolvent of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and analyticity of the underlying semigroup . . . 171
5.6 Eigenvalue problems . . . 173
5.6.1 Direct eigenvalue problem . . . 173
5.6.2 Adjoint eigenvalue problem . . . 174
5.7 Stabilization of the linearized system . . . 175
5.7.1 Stabilizability of $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ . . . 178
5.7.2 Feedback control law . . . 180
5.8 Stabilization of the nonhomogeneous linearized system . . . 181
5.9 Stabilization of the nonlinear system . . . 188
5.9.1 Estimate of nonlinear terms . . . 188
5.9.2 Proof of Theorem 5.2.1 . . . 192
Appendix A: Nonlinear terms coming from the geometric transformation (5.2.8) . . . 195
Appendix B: Proof of Theorem 5.3.2 . . . 197
Appendix C: Justification of the identity (5.4.14) (formal computations) . . . 199
Appendix D: Proof of Proposition 5.7.1 . . . 204
Appendix E: Proof of Proposition 5.7.3 . . . 206

6 Numerical simulations of the stabilization problem of the fluid-structure interaction system 209
6.1 Introduction . . . 210
6.2 Stabilization of the semi-discrete system . . . 213
6.2.1 Semi-discrete approximation . . . 214
6.2.2 Eliminating the fluid domain variables $\overline{\mathbf{W}}$ and $\overline{\mathbf{D}}$ . . . 216
6.2.3 Finite dimensional controlled system . . . 217
6.2.4 Spectral decomposition of the operators . . . 217
6.2.5 The projected dynamical system and computation of the feedback . . . 219
6.3 Algorithm for FSI direct problem . . . 220
6.4 Numerical experiments . . . 221
6.4.1 Case 1: Rigidity coefficient $\boldsymbol{\alpha = 1}$, $Re = 200$ and $\gamma = 10^{-6}$ . . . 223
6.4.2 Case 2: Rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, $Re = 200$ and $\gamma = 10^{-6}$ . . . 228
6.4.3 Case 3: Rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, $Re = 200$ and $\gamma = 10^{-6}$ . . . 233

Bibliographie 240

Chapitre 1

Introduction

Sommaire du présent chapitre

1.1 Contexte général **1**

1.2 Description du modèle étudié **2**

1.3 Résultats de la thèse **5**

1.3.1 Chapitre 2 : Étude du système de Stokes stationnaire avec conditions aux limites mixtes dans des domaines polygonaux curvilignes non convexes 5

1.3.2 Chapitre 3 : Existence d'une solution forte locale en temps pour le système d'interaction fluide-structure 10

1.3.3 Chapitre 4 : Simulations numériques du système d'interaction fluide-structure 16

1.3.4 Chapitre 5 : Stabilisation du système d'interaction fluide-structure 22

1.3.5 Chapitre 6 : Simulations numériques du problème de stabilisation du système d'interaction fluide-structure 30

1.4 Perspectives **36**

1.1 Contexte général

Le phénomène d'interaction fluide-structure est présent dans différents domaines, comme par exemple la biologie ou l'aéronautique. Plus spécifiquement, un exemple concret dans le domaine de la biologie est le cas de l'écoulement sanguin dans une artère, ou dans le cas de l'aéronautique, l'écoulement d'air autour d'une aile d'avion.

Parmi la grande diversité de modèles fluide-structure existant dans la littérature, nous nous intéressons, dans ce qui suit, à un système couplant les équations de Navier-Stokes avec des conditions aux limites mixtes comme modèle pour le fluide, et une équation d'Euler-Bernoulli amortie, avec des conditions limites de type encastrée et extrémité libre, comme modèle pour la structure. Nous présentons ci-dessous un bref résumé du contenu de chaque chapitre de la thèse.

• **Existence de solutions** (Chapitres 2 et 3)

Dans le Chapitre 2, nous analysons le système de Stokes stationnaire avec conditions aux limites mixtes dans des domaines présentant des coins rentrants. Ce chapitre est utilisé dans l'analyse du Chapitre 3, où nous démontrons un résultat d'existence d'une solution forte locale en temps du système fluide-structure.

• **Stabilisation du système** (Chapitre 5)

Dans le Chapitre 5, nous nous intéressons à la stabilisation du modèle fluide-structure autour d'une solution stationnaire instable. Afin d'atteindre cet objectif, nous construisons un contrôle de dimension finie agissant sur la structure.

• **Simulations numériques** (Chapitres 4 et 6)

Un troisième axe de ce travail de thèse, concerne les simulations numériques du système en question. Dans une première partie, au Chapitre 4, nous présentons des simulations numériques du problème direct. Ensuite, au Chapitre 6, nous présentons des simulations du problème de stabilisation.

1.2 Description du modèle étudié

Nous supposons qu'une structure $\mathcal{S} = \mathcal{S}_r \cup \mathcal{S}_e$ est immergée dans une cavité remplie d'un fluide Newtonien incompressible et visqueux en deux dimensions. Ici, $\mathcal{S}_r$ designe la partie rigide de la structure et $\mathcal{S}_e$ la partie élastique. Nous désignerons par Ω le domaine représentant la configuration de référence, lequel est donné par

$$\Omega = ([-L/2, L] \times [-\ell, \ell]) \setminus \mathcal{S}, \tag{1.2.1}$$

comme indiqué à la Figure 1.1.

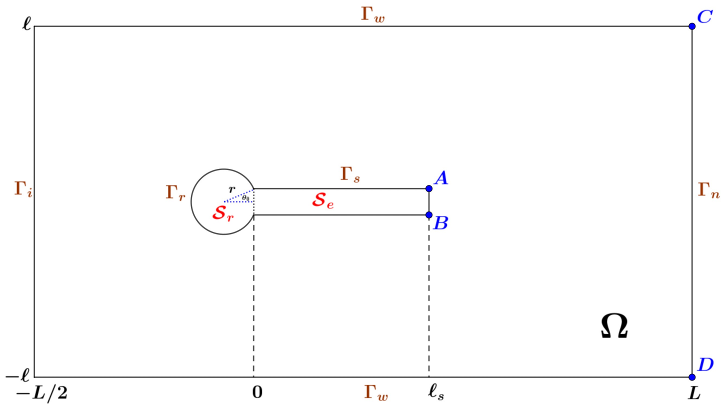


FIGURE 1.1 – Configuration de référence.

La frontière du domaine Ω est donnée par

$$\Gamma = \Gamma_i \cup \Gamma_r \cup \Gamma_s \cup \Gamma_w \cup \Gamma_n, \tag{1.2.2}$$

où $\Gamma_s = \Gamma_s^- \cup \Gamma_s^+ \cup \Gamma_s^\ell$, avec $\Gamma_s^- = [0, \ell_s] \times \{-e\}$, $\Gamma_s^+ = [0, \ell_s] \times \{e\}$ et $\Gamma_s^\ell = \{\ell_s\} \times [-e, e]$. Nous posons également $\Gamma_d = \Gamma \setminus \Gamma_n$.

Dans le cas instationnaire, l'interaction entre le fluide et la structure entraîne une déformation de la géometrie, comme le montre la Figure 1.2.

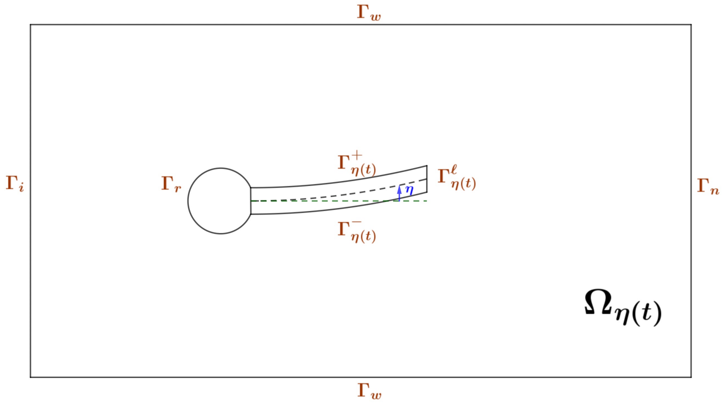


FIGURE 1.2 – Domaine physique. Les lignes pointillées vertes indiquent la ligne centrale de référence.

Soit $T > 0$. Nous supposons que la partie élastique $\mathcal{S}_e$ de la structure est entièrement décrite par le déplacement η de la ligne centrale (voir Figure 1.2). Pour une fonction η donnée, définie de $(0, T) \times (0, \ell_s)$ dans $\mathbb{R}$, qui décrit le déplacement de l'axe central de la poutre, nous désignons par $\Omega_{\eta(t)}$ le domaine du fluide à l'instant t et par $\Gamma_{\eta(t)} = \Gamma_{\eta(t)}^{-} \cup \Gamma_{\eta(t)}^{+} \cup \Gamma_{\eta(t)}^{\ell}$ l'interface fluide-structure, où $\Gamma_{\eta(t)}^{-}$ et $\Gamma_{\eta(t)}^{+}$ représentent respectivement le bas et le haut de la structure, et $\Gamma_{\eta(t)}^{\ell}$ la partie latérale (voir Figure 1.2). Plus précisément, $\Gamma_{\eta(t)}^{-}$, $\Gamma_{\eta(t)}^{+}$ et $\Gamma_{\eta(t)}^{\ell}$ sont données par

$$\begin{aligned} &\Gamma_{\eta(t)}^{-} = \{(x_1, \eta(t, x_1) - e) |\, x_1 \in [0, \ell_s]\},\ \Gamma_{\eta(t)}^{+} = \{(x_1, \eta(t, x_1) + e) |\, x_1 \in [0, \ell_s]\} \\ &\text{et } \Gamma_{\eta(t)}^{\ell} = \{(\ell_s, x_2) \mid x_2 = (1-\lambda)(-e + \eta(t, \ell_s)) + \lambda(e + \eta(t, \ell_s))),\ \ \lambda \in [0, 1]\}. \end{aligned} \tag{1.2.3}$$

Pour $0 < T \leq \infty$, nous notons

$$\begin{aligned} Q_\eta^T &= \bigcup_{t \in (0,T)} \Big(\{t\} \times \Omega_{\eta(t)}\Big),\ \Sigma_\eta^T = \bigcup_{t \in (0,T)} \Big(\{t\} \times \Gamma_{\eta(t)}\Big), \\ Q^T &= (0, T) \times \Omega,\ \ \Sigma_s^T = (0, T) \times \Gamma_s, \\ \Sigma_i^T &= (0, T) \times \Gamma_i,\ \ \Sigma_w^T = (0, T) \times \Gamma_w, \\ \Sigma_r^T &= (0, T) \times \Gamma_r,\ \ \Sigma_n^T = (0, T) \times \Gamma_n. \end{aligned} \tag{1.2.4}$$

• **Le fluide**

Nous supposerons que le fluide est Newtonien, incompressible et visqueux en deux dimensions. Nous modélisons le fluide par les équations de Navier-Stokes incompressibles

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, p) = 0 \ \text{ dans } Q_\eta^T, \tag{1.2.5a}$$

$$\operatorname{div} \mathbf{u} = 0 \ \text{ dans } Q_\eta^T, \tag{1.2.5b}$$

$$\mathbf{u}(0) = \mathbf{u}^0 \ \text{ dans } \Omega, \tag{1.2.5c}$$

où $\mathbf{u}$ et p sont respectivement la vitesse et la pression du fluide. Ici, $\sigma(\mathbf{u}, p)$ représente le tenseur des contraintes du fluide donné par

$$\sigma(\mathbf{u}, p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

avec $\nu > 0$ représentant la viscosité du fluide.

Dans cette thèse, nous imposons une condition de type Dirichlet non homogène à l'entrée Γ_i

du canal, tandis qu'à la sortie Γ_n, nous imposons une condition de type Neumann homogène. D'autre part, sur les parties supérieure et inférieure Γ_w du canal, ainsi que sur Γ_r, nous imposons une condition de type Dirichlet homogène. Plus précisément,

$$\mathbf{u} = \mathbf{g}^i \;\text{ sur } \Sigma_i^T, \;\; \sigma(\mathbf{u}, p)\mathbf{n} = 0 \;\text{ sur } \Sigma_n^T, \tag{1.2.6a}$$

$$\mathbf{u} = 0 \;\text{ sur } \Sigma_w^T \cup \Sigma_r^T, \tag{1.2.6b}$$

où $\mathbf{g}^i$ est donnée.

- **La structure**

Nous modélisons le déplacement η de la ligne centrale par l'équation de poutre Euler-Bernoulli amortie

$$\partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u}, p, \eta) \text{ dans } (0, T) \times (0, \ell_s), \tag{1.2.7a}$$

$$\eta(0) = 0 \;\text{ et }\; \partial_t\eta(0) = \eta_2^0 \;\text{ dans } (0, \ell_s), \tag{1.2.7b}$$

où les paramètres $\alpha > 0$ et $\gamma > 0$ sont des constantes relatives à la nature de la structure. Ici, $\Delta_s^2 = \frac{\partial^4}{\partial x^4}$ avec comme domaine $\mathcal{D}(\Delta_s^2) = H^4_{\{0,\ell_s\}}(0, \ell_s)$, représente l'opérateur bi-Laplacien sur $(0, \ell_s)$. Dans cette thèse, l'opérateur d'amortissement B sera défini comme une puissance de Δ_s^2, sur un domaine $\mathcal{D}(B)$ ad hoc. Plus précisément, $B = (\Delta_s^2)^r$, avec $1/2 \leq r \leq 1$. Le terme source H dans le second membre de l'équation de la structure est donné par

$$H(\mathbf{u}, p, \eta) = -\left(\sigma^+(\mathbf{u}, p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u}, p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1 + (\partial_{x_1}\eta)^2} \cdot \vec{\mathbf{e}}_2, \tag{1.2.8}$$

où

$$\sigma^{\pm}(\mathbf{u}, p) = \sigma(\mathbf{u}(t, x_1, \eta(t, x_1) \pm e), p(t, x_1, \eta(t, x_1) \pm e)),$$

et $\mathbf{n}^+_{\eta(t)}$ (respectivement $\mathbf{n}^-_{\eta(t)}$) est le vecteur unitaire normal à $\Gamma^+_{\eta(t)}$ (respectivement à $\Gamma^-_{\eta(t)}$) extérieur à $\Omega_{\eta(t)}$. Ici, $\vec{\mathbf{e}}_2 = (0, 1)$. La déduction de l'expression de H donnée en (1.2.8), ainsi que les simplifications effectuées dans le modèle, sont présentées dans l'Annexe A.

À l'extrémité gauche de la structure en $x = 0$, nous imposons des conditions d'encastrement et une condition d'extremité libre en $x = \ell_s$. De manière plus précise,

$$\eta = 0 \;\text{ et }\; \partial_{x_1}\eta = 0 \;\text{ sur } (0, T) \times \{0\}, \tag{1.2.9a}$$

$$\partial^2_{x_1}\eta = 0 \;\text{ et }\; \partial^3_{x_1}\eta = 0 \;\text{ sur } (0, T) \times \{\ell_s\}. \tag{1.2.9b}$$

- **Les interactions**

Tout d'abord, nous considérons la condition cinématique

$$\mathbf{u} = \eta_t\vec{\mathbf{e}}_2 \;\text{ on } \Sigma_\eta^T. \tag{1.2.10}$$

La condition dynamique est encapsulée dans le terme H qui apparaît dans le membre de droite de l'équation (1.2.7a), dont l'expression explicite est donnée en (1.2.8).

En résumé, le modèle fluide-structure qui nous intéresse dans cette thèse est donné par

$$\begin{cases}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ dans } Q_\eta^T,\\
\operatorname{div}\mathbf{u} = 0 \ \text{ dans } Q_\eta^T,\\
\mathbf{u} = \mathbf{g}^i \ \text{ sur } \Sigma_i^T, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma_r^T \cup \Sigma_w^T,\\
\mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_\eta^T, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ sur } \Sigma_n^T,\\
\mathbf{u}(0) = \mathbf{u}^0 \ \text{ dans } \Omega,\\
\partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) \text{ dans } (0,T)\times(0,\ell_s),\\
\eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ sur } (0,T)\times\{0\},\\
\partial_{x_1}^2\eta = 0 \ \text{ and } \ \partial_{x_1}^3\eta = 0 \ \text{ sur } (0,T)\times\{\ell_s\},\\
\eta(0) = 0 \ \text{ and } \ \partial_t\eta(0) = \eta_2^0 \ \text{ dans } (0,\ell_s).
\end{cases} \tag{1.2.11}$$

Dans ce qui suit, nous décrirons plus en détail la contribution de chaque chapitre de la thèse.

1.3 Résultats de la thèse

1.3.1 Chapitre 2 : Étude du système de Stokes stationnaire avec conditions aux limites mixtes dans des domaines polygonaux curvilignes non convexes

• Formulation du problème et littérature existante

Dans ce chapitre, nous considérons un système de Stokes stationnaire avec des conditions aux limites mixtes, de type Dirichlet et Neumann, dans un domaine Ω polygonal curviligne non convexe borné de $\mathbb{R}^2$ (par exemple, le domaine représenté dans la Figure 1.1). Plus précisément, nous nous intéressons à l'étude du système

$$\begin{cases}
-\operatorname{div}\sigma(\mathbf{w},\pi) = \mathbf{F} \ \text{ dans } \Omega, \ \ \operatorname{div}\mathbf{w} = h \ \text{ dans } \Omega,\\
\mathbf{w} = \mathbf{g} \ \text{ sur } \Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n} = 0 \ \text{ sur } \Gamma_n,
\end{cases} \tag{1.3.1}$$

où $\mathbf{F}, h$ et $\mathbf{g}$ sont données.

Bien que les résultats énoncés dans ce chapitre soient intéressants en eux-mêmes, leur motivation principale provient de l'étude de l'existence de solutions fortes du système d'interaction fluide-structure (1.2.11), problème dans lequel nous devons étudier la régularité des solutions du système (1.3.1), lorsque le domaine est donné, par exemple, par celui représenté sur Figure 1.2, et que les donneés $\mathbf{F}$ et h sont irrégulières. Pour entamer la discussion, commençons par rappeler les résultats existants dans la littérature.

D'après [MR10, Theorem 9.4.5], nous savons que la solution $(\mathbf{w},\pi)$ du système de Stokes (1.3.1) avec des conditions aux limites mixtes Dirichlet-Neumann satisfait

$$\begin{aligned}
&(\mathbf{w},\pi) \text{ appartient à l'espace de Sobolev pondéré } \mathbf{H}_\delta^2(\Omega)\times H_\delta^1(\Omega)\\
&\text{lorsque } \mathbf{F}\in\mathbf{L}_\delta^2(\Omega), \ h\in H_\delta^1(\Omega) \text{ et } \mathbf{g}\in\mathbf{H}_\delta^{\frac{3}{2}}(\Gamma_d),
\end{aligned} \tag{R1}$$

pour tout $\delta\in(\delta^*,1)$, où $\delta^*\in(0,1/2)$.
D'autre part, pour le système de Stokes avec seulement une condition limite de type Dirichlet, $\mathbf{w}=\mathbf{g}$ sur $\Gamma=\Gamma_d$ (i.e., $\Gamma_n=\emptyset$), plus une condition de compatibilité appropriée sur h et $\mathbf{g}$, nous

avons

$$\begin{aligned}&(\mathbf{w},\pi) \text{ appartient à l'espace de Sobolev } \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)\times H^{\frac{1}{2}+\alpha}(\Omega),\\ &\text{lorsque } \mathbf{F}\in\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\ h\in H^{\frac{1}{2}+\alpha}(\Omega),\ \mathbf{g}\in\mathbf{H}^{1+\alpha}(\Gamma) \text{ pour tout } \alpha\in(0,\alpha^*),\end{aligned} \tag{R2}$$

où l'exposant critique $\alpha^*\in(0,1/2)$ dépend des angles aux coins de Γ. Ce résultat est démontré dans [Dau89, Theorem 5.5(a)] lorsque $\mathbf{g}=0$ et étendu à $\mathbf{g}\neq 0$ dans [BR, Corollary 3.3].

Un système d'interaction fluide-structure similaire à (1.2.11) a été étudié dans [FNR19], mais avec des conditions d'encastrement aux deux extrémités de la poutre élastique. Dans ce cas, le problème non linéaire peut être étudié dans le cadre des espaces avec poids $\mathbf{H}^2_\delta(\Omega)$ et $H^1_\delta(\Omega)$. Par conséquent, la question naturelle est de savoir s'il est possible d'utiliser le résultat de régularité (R1) dans le cas du modèle (1.2.11).

Dans le cas considéré du modèle (1.2.11), où une condition d'encastrement est imposée d'un côté et une condition d'extrémité libre de l'autre, le résultat de régularité (R1) reste toujours valable pour le système d'interaction fluide-structure linéarisé. Cependant, cette approache n'est pas suffisante pour traiter le système d'interaction fluide-structure non linéaire. Cela est dû à un décalage entre les résultats de régularité obtenus dans les espaces de Sobolev avec poids pour les solutions du modèle linéaire non homogène et la régularité correspondante de certains termes non linéaires du modèle (qui joueront le rôle des termes non homogènes dans une procédure de point fixe).

• **Présentation de nos résultats**

Afin de simplifier la présentation de nos résultats, nous considérons la géometrie Ω donnée à la Figure 1.1. Les hypothèses générales sur la géometrie sont présentées dans la sous-section 2.2.1. Nous introduissons les espaces fonctionnels suivants :

$$\begin{aligned}&\mathbf{L}^2(\Omega)=L^2(\Omega;\mathbb{R}^2) \text{ et, avec } s>0,\ \mathbf{H}^s(\Omega)=H^s(\Omega;\mathbb{R}^2),\\ &\mathbf{H}^1_{\Gamma_d}(\Omega)=\{\mathbf{u}\in\mathbf{H}^1(\Omega)\mid \mathbf{u}=0 \ \text{ sur } \Gamma_d\},\\ &\mathbf{V}^0_{n,\Gamma_d}(\Omega)=\{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid \operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \ \mathbf{u}\cdot\mathbf{n}=0 \text{ sur } \Gamma_d\},\\ &\mathbf{V}^1_{\Gamma_d}(\Omega)=\mathbf{H}^1_{\Gamma_d}(\Omega)\cap\mathbf{V}^0_{n,\Gamma_d}(\Omega).\end{aligned}$$

Étant donnée $\beta>0$, nous introduisons les espaces de Sobolev avec poids

$$\begin{aligned}\|\mathbf{w}\|_{\mathbf{H}^2_\beta} &:= \left(\sum_{|k|=0}^{2}\sum_{i=1}^{2}\int_\Omega\Big(\prod_{J\in\mathcal{J}} r_J^{2\beta}\Big)|\partial^k w_i|^2dx\right)^{1/2}, \quad \mathbf{w}\in C^\infty(\overline{\Omega};\mathbb{R}^2)\\ \|p\|_{H^1_\beta} &:= \left(\sum_{|k|=0}^{1}\int_\Omega\Big(\prod_{J\in\mathcal{J}} r_J^{2\beta}\Big)|\partial^k p|^2dx\right)^{1/2}, \quad p\in C^\infty(\overline{\Omega};\mathbb{R})\end{aligned} \tag{1.3.2}$$

où r_J designe la distance au point de junction $J\in\mathcal{J}$, $k=(k_1,k_2)\in\mathbb{N}^2$ est un muti-indice de longueur $|k|=k_1+k_2$, ∂^k désigne l'opérateur différentiel partiel correspondant et $\mathbf{w}=(w_1,w_2)$. Nous désignons par $\mathbf{H}^2_\beta(\Omega;\mathbb{R}^2)$ (respectivement $H^1_\beta(\Omega)$) la fermeture de $C^\infty(\overline{\Omega};\mathbb{R}^2)$ (respectivement $C^\infty(\overline{\Omega})$) dans la norme $\|\cdot\|_{\mathbf{H}^2_\beta}$ (respectivement $\|\cdot\|_{H^1_\beta}$).

On désigne par $\mathcal{J}_{dd}\subset\mathcal{J}$ l'ensemble des points correspondant à une jonction entre deux conditions aux limites de Dirichlet. Soient $\mathcal{U}$ et $\mathcal{V}$ deux ouverts disjoints de $\mathbb{R}^2$ tels que $\mathcal{J}_{d,d}\subset\mathcal{U}$ et $\Gamma_n\subset\mathcal{V}$. En particulier, $\mathcal{V}$ ne contient aucun point correspondant à une jonction Dirichlet-Dirichlet. Introduisons maintenant une fonction de troncature $\Psi\in\mathcal{C}^\infty(\mathbb{R}^2)$ vérifiant $0\leq\Psi\leq 1$

et

$$\Psi = 1 \text{ dans } \mathcal{U} \text{ et } \Psi = 0 \text{ dans } \mathcal{V}. \tag{1.3.3}$$

Soit $\alpha \in (0, \alpha^*)$ et $\delta \in (\delta^*, 1)$. Nous introduisons les espaces de Sobolev hétérogènes suivants

$$\begin{aligned}
\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) &= \left\{ \mathbf{F} \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1(\Omega) \right\}, \\
H_\delta^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H_\delta^1(\Omega) \right\}, \\
\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) &= \left\{ \mathbf{v} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{v} \in \mathbf{H}_\delta^2(\Omega) \right\},
\end{aligned} \tag{1.3.4}$$

qui sont respectivement munis des normes

$$\begin{aligned}
\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}} &:= \left(\|\mathbf{F}\|^2_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}} + \|(1-\Psi)\mathbf{F}\|^2_{\mathbf{L}^2} \right)^{1/2}, \\
\|p\|_{H^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H^1} \right)^{1/2}, \\
\|p\|_{H_\delta^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H_\delta^1} \right)^{1/2}, \\
\|\mathbf{u}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}} &:= \left(\|\mathbf{u}\|^2_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|(1-\Psi)\mathbf{u}\|^2_{\mathbf{H}_\delta^2} \right)^{1/2}.
\end{aligned}$$

Présentons maintenant les principaux résultats de ce chapitre, ainsi que les idées essentielles de leurs preuves respectives.

1. **Régularité du système de Stokes stationnaire**.

Theorem 1.3.1. *Supposons que* $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$*,* $h \in H^{\frac{1}{2}+\alpha,1}(\Omega)$ *et* $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$*. Alors, la solution variationnelle* $(\mathbf{w}, \pi)$ *du système* (1.3.1) *appartient à* $\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$*. De plus, il existe une constante* $C_{\alpha,\delta} > 0$ *telle que*

$$\|\mathbf{w}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)} + \|\pi\|_{H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)} \leq C_{\alpha,\delta} \left(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} + \|h\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_d)} \right). \tag{1.3.5}$$

Idée de la preuve. L'idée principale de la preuve est d'utiliser un argument de troncature qui nous permet de considérer deux systèmes : un premier système avec des conditions de type Dirichlet, c'est-à-dire $\Gamma_n = \emptyset$, qui nous permet d'utiliser le résultat de régularité (R2). Ensuite, un second système avec des conditions mixtes, nous permettant d'utiliser le résultat de régularité (R1). Il est important de souligner que la fonction de troncature Ψ choisie est la même que celle définie dans (1.3.3), ce qui est naturel, compte tenu de la régularité des données $\mathbf{F}$ et h.

2. **Analyticité du semigroupe associé à l'opérateur de Stokes sur** $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$.

Commençons par introduire l'opérateur de Stokes. L'opérateur de Stokes $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ dans $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ est défini par

$$\begin{aligned}
\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)) = \Big\{ \mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \cap \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists \pi \in H^{\frac{1}{2}+\alpha}(\Omega) \text{ tel que} \\
\operatorname{div} \sigma(\mathbf{w}, \pi) \in \mathbf{L}^2(\Omega) \text{ et } \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \text{ sur } \Gamma_n \Big\},
\end{aligned}$$

$$A_0 \mathbf{w} = P \operatorname{div} \sigma(\mathbf{w}, \pi).$$

Pour tout $\alpha \in (0, \alpha^*)$, nous introduisons l'espace hétérogène

$$\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega) := \left\{ \mathbf{v} \in \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega) \mid (1-\Psi)\mathbf{v} \in \mathbf{L}^2(\Omega) \right\}. \tag{1.3.6}$$

Cet espace muni de sa norme naturelle est un espace de Hilbert (voir Proposition 2.4.5).

Comme l'opérateur A_0 est un isomoprhisme de $\mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^0(\Omega))$ dans $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$ et de $\mathbf{V}_{\Gamma_d}^1(\Omega)$ dans $\mathbf{V}_{\Gamma_d}^{-1}(\Omega)$, on en déduit que qu'il est également un isomorphisme de

$$\mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)) = [\mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^0(\Omega)), \mathbf{V}_{\Gamma_d}^1(\Omega)]_{\frac{1}{2}-\alpha} \text{ dans } [\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{\frac{1}{2}-\alpha} = \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega). \tag{1.3.7}$$

L'égalité $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{\frac{1}{2}-\alpha} = \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$ est établie dans le Lemme 2.4.7, dont la preuve repose sur un résultat similaire établi dans [MM08] dans le cas de conditions de Dirichlet partout et sur l'existence d'un opérateur d'extension E qui est précisé en (**??**). Dans le Lemme 3.3.1 du Chapitre 3, nous montrons l'existence d'un tel opérateur dans le cadre de la geométrie étudiée dans ce chapitre.

Pour tout $\theta \in (\pi/2, \pi)$, nous définissons le secteur Σ_θ par

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \theta\}.$$

Theorem 1.3.2. *L'opérateur non borné $(A_0, \mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)))$ est le générateur infinitésimal d'un semigroupe analytique sur $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$.*

Idée de la preuve. D'après [EN06, Théorème 4.6, p. 95], il suffit de montrer que l'opérateur $(A_0, \mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)))$ est sectoriel et à domaine dense.

(a) *Sectorialité de l'opérateur de Stokes sur $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$*

Proposition 1.3.1. *Soit $\alpha \in (0, \alpha^*)$. Il existe $\theta_0 \in (\pi/2, \pi)$ et $C > 0$ tel que*

$$\|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))} \leq \frac{C}{|\lambda|}, \quad \textit{pour tout } \lambda \in \Sigma_{\theta_0} \setminus \{0\}. \tag{1.3.8}$$

Idée de la preuve. L'inégalité (1.3.8) s'obtient par interpolation. D'abord, comme l'opérateur de Stokes $(A_0, \mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^0(\Omega)))$ est sectoriel sur $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$, il existe $\theta_0 \in (\pi/2, \pi)$ et $C_0 > 0$ tel que

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^0(\Omega)} \leq \frac{C_0}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^0(\Omega)}, \tag{1.3.9}$$

pour tout $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$ et $\mathbf{F} \in \mathbf{V}_{n,\Gamma_d}^0(\Omega)$. D'autre part, des calculs directs permettent de montrer qu'il existe $C_1 > 0$ tel que

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}_{\Gamma_d}^{-1}} \leq \frac{C_1}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}_{\Gamma_d}^{-1}}, \tag{1.3.10}$$

pour tout $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$ et $\mathbf{F} \in \mathbf{V}_{n,\Gamma_d}^0(\Omega)$. Ensuite, en interpolant (1.3.9) et (1.3.10) et en utilisant le Lemme 2.4.7 et la densité de $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$ dans $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, on déduit (1.3.8) pour tout $\mathbf{F} \in \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega)$ et pour tout $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$.

(b) *L'opérateur* $(A_0, \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ *est à domaine dense.*

Proposition 1.3.2. *Le domaine* $\mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega))$ *de l'opérateur de Stokes* $(A_0, \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ *est dense dans* $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.

Idée de la preuve. La preuve repose sur une modification de [Paz83, Théorème 4.6, p. 16] (voir Proposition 2.4.9(iii)) et de l'estimation (1.3.8).

3. **Réécriture du système (1.3.1) en termes d'opérateurs**.

Avant de présenter le théorème principal de cette sous-section, nous introduirons des notations. D designe un opérateur de relèvement de la trace sur Γ_i (voir la définition (2.4.64)), tandis que les opérateurs N_p et N_v sont définis comme suit :

$$N_p \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega), H^{\frac{1}{2}+\alpha,1}(\Omega)), \quad N_p\mathbf{F} = q, \tag{1.3.11}$$

où q est solution du système

$$\Delta q = \operatorname{div}\mathbf{F} \ \text{ dans } \Omega, \quad \frac{\partial q}{\partial \mathbf{n}} = \mathbf{F}\cdot\mathbf{n} \ \text{ sur } \Gamma_d, \quad q = 0 \ \text{ sur } \Gamma_n, \tag{1.3.12}$$

et

$$N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega), L^2(\Omega)), \quad N_v\mathbf{w} = \rho, \tag{1.3.13}$$

où ρ est solution du système

$$\Delta \rho = 0 \ \text{ dans } \Omega, \quad \frac{\partial \rho}{\partial \mathbf{n}} = \nu\Delta\mathbf{w}\cdot\mathbf{n} \ \text{ sur } \Gamma_d, \quad \rho = 2\nu\varepsilon(\mathbf{w})\mathbf{n}\cdot\mathbf{n} \ \text{ sur } \Gamma_n. \tag{1.3.14}$$

Theorem 1.3.3. *Supposons que* $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$ *et* $h = 0$. *Le couple* $(\mathbf{w}, \pi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega) \times H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)$ *est une solution variationnelle de* (1.3.1) *si et seulement si* $P\mathbf{w}$, $(I-P)\mathbf{w}$, *et* π *sont solutions du système suivant :*

$$\begin{cases} -A_0P\mathbf{w} + A_0PD\mathbf{g} = P\mathbf{F}, \\ (I-P)\mathbf{w} = (I-P)D\mathbf{g}, \quad \pi = N_p\mathbf{F} + N_v\mathbf{w}. \end{cases} \tag{1.3.15}$$

Idée de la preuve. Les principales difficultés de la preuve résident dans le fait de montrer que les opérateurs $N_p \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega), H^{\frac{1}{2}+\alpha,1}(\Omega))$ et $N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega), L^2(\Omega))$ sont bien définis et que la pression peut s'exprimer comme $\pi = N_p\mathbf{F} + N_v\mathbf{w}$.

(a) *Opérateurs N_p et N_v bien définis*

Pour montrer que les opérateurs N_p et N_v sont bien définis, il est tout d'abord nécessaire de préciser le sens des solutions de (1.3.12) et (1.3.14), respectivement. Dans le cas du système (1.3.12), où l'on utilise la notion de solution par transposition, une analyse précise de la régularité de celle-ci est requise (voir Lemme 2.4.4). En ce qui concerne le système (1.3.14), où l'on utilise une notion de solution très-faible, on conclut à l'aide du Théorème de représentation de Riesz (voir Lemme 2.4.8).

(b) *Justification de l'expression de la pression π*

Une fois les opérateurs N_p et N_v bien définis, il est nécessaire de montrer que la pression peut s'exprimer comme $\pi = N_p\mathbf{F} + N_v\mathbf{w}$. L'idée centrale de la démonstration repose sur l'utilisation du système satisfait par $(\mathbf{w}, \pi)$ ainsi qu'un argument de densité (voir Lemme 2.4.9). Ce dernier argument est utilisé pour justifier certaines intégrations par parties.

1.3.2 Chapitre 3 : Existence d'une solution forte locale en temps pour le système d'interaction fluide-structure

Tout au long de cette section, nous considérons la configuration geométrique représentée dans la Figure 1.2, dont la description précise a été introduite dans la Section 1.2 (voir (1.2.1), (1.2.2) et (1.2.4)).

• **Formulation du problème et littérature existante**

Dans ce chapitre, nous prouvons l'existence d'une solution forte locale en temps pour le système d'interaction fluide-structure

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ dans } Q_\eta^T, \tag{1.3.16a}$$
$$\operatorname{div}\mathbf{u} = 0 \ \text{ dans } Q_\eta^T, \tag{1.3.16b}$$
$$\mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma_i^T, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma_w^T, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma_r^T, \tag{1.3.16c}$$
$$\mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_\eta^T, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ sur } \Sigma_n^T, \tag{1.3.16d}$$
$$\mathbf{u}(0) = \mathbf{u}^0 \ \text{ dans } \Omega, \tag{1.3.16e}$$
$$\partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) \text{ dans } (0,T)\times(0,\ell_s), \tag{1.3.16f}$$
$$\eta = 0 \ \text{ et } \ \partial_{x_1}\eta = 0 \ \text{ sur } (0,T)\times\{0\}, \tag{1.3.16g}$$
$$\partial_{x_1}^2\eta = 0 \ \text{ et } \ \partial_{x_1}^3\eta = 0 \ \text{ sur } (0,T)\times\{\ell_s\}, \tag{1.3.16h}$$
$$\eta(0) = 0 \ \text{ et } \ \partial_t\eta(0) = \eta_2^0 \ \text{ dans } (0,\ell_s), \tag{1.3.16i}$$

où

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}_{\eta(t)}^+ + \sigma^-(\mathbf{u},p)\mathbf{n}_{\eta(t)}^-\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2,$$

avec

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x_1,\eta(t,x_1)\pm e), p(t,x_1,\eta(t,x_1)\pm e)),$$

et $\mathbf{n}_{\eta(t)}^+$ (respectivement $\mathbf{n}_{\eta(t)}^-$) est le vecteur normal unitaire $\Gamma_{\eta(t)}^+$ (respectivement $\Gamma_{\eta(t)}^-$) extérieur à $\Omega_{\eta(t)}$. Ici, $\vec{\mathbf{e}}_2 = (0,1)$. Tout au long de ce chapitre, nous supposerons que l'opérateur d'amortissement B est donné par $B = (\Delta_s^2)^{1/2}$ avec un domaine $\mathcal{D}(B)$ que nous préciserons ultérieurement. Cependant, comme nous le verrons dans la Remarque 6, le résultat principal de ce chapitre reste également valable dans le cas $B = (\Delta_s^2)^r$ avec $r \in (1/2, 1]$, où le domaine $\mathcal{D}(B)$ sera précisé plus tard.

Nous présentons ci-dessous une liste non-exhaustive de travaux portant sur l'existence de solutions pour des modèles d'interaction fluide-structure. Nous limiterons cette présentation à l'étude de modèles couplant les équations de Navier-Stokes incompressibles et une structure modélisée par une équation de poutre ou de plaque.

En ce qui concerne l'étude de solutions faibles, à notre connaisance, le premier résultat dans cette direction est dû à Chambolle et al. [CDEG05], où les auteurs ont étudié un système couplant les équations incompressible de Navier-Stokes en $3D$ et un équation de plaque amortie $2D$. Ce résultat a ensuite été étendu par Grandmont [Gra08] au cas non amorti. Dans les deux travaux, qui sont également valables dans le cas $2D/1D$, l'existence de solutions faibles se vérifie tant que la structure ne touche pas la partie fixe de la frontière du fluide. Récemment, Casanova et al. [CGH21] ont démontré, dans un cadre $2D/1D$ sans amortissement, l'existence de solutions faibles globales en temps, indépendamment d'un éventuel contact entre la structure et la partie fixe du domaine fluide.

Concernant l'étude de solutions fortes, à notre connaissance, le premier résultat a été obtenu par Beirão da Veiga [Bei04] dans le cas où la dynamique de la structure est régie par une équation de poutre d'Euler-Bernoulli amortie. Plus précisément, l'auteur montre un résultat d'existence locale en temps sous l'hypothèse d'une condition de petitesse sur les données initiales. Ce résultat a ensuite été amelioré par Lequeurre [Leq11], qui démontre le même résultat d'existence locale en temps, mais sans supposer une condition de petitesse sur les données initiales. Le cas non amorti est traité dans Badra et al. [BT19], où les auteurs prouvent l'existence d'une solution forte locale en temps.

• **Présentation de nos résultats**

La principale contribution de ce chapitre est la preuve d'existence d'une solution forte locale en temps du système (1.3.16). Commençons par introduire les espaces fonctionnels :

$$\begin{aligned}
&\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2) \text{ et, avec } s>0,\ \mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2),\\
&\mathbf{H}^s_{\Gamma_d}(\Omega) = \{\mathbf{u}\in\mathbf{H}^s(\Omega) \mid \mathbf{u}=0 \text{ sur } \Gamma_d\} \ \text{ avec } s>1/2,\\
&\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{u}\in\mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \ \mathbf{u}\cdot\mathbf{n}=0 \text{ sur } \Gamma_d\},\\
&\mathbf{V}^1_{\Gamma_d}(\Omega) = \mathbf{H}^1_{\Gamma_d}(\Omega)\cap\mathbf{V}^0_{n,\Gamma_d}(\Omega),\\
&H^1_{\{0\}}(0,\ell_s) = \{\mu\in H^1(0,\ell_s) \mid \mu(0)=0\},\\
&H^2_{\{0\}}(0,\ell_s) = \{\mu\in H^2(0,\ell_s) \mid \mu(0)=\partial_{x_1}\mu(0)=0\},\\
&H^3_{\{0,\ell_s\}}(0,\ell_s) = \{\mu\in H^3(0,\ell_s)\cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s)=0\},\\
&H^4_{\{0,\ell_s\}}(0,\ell_s) = \{\mu\in H^4(0,\ell_s)\cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s)=\partial^3_{x_1}\mu(\ell_s)=0\},\\
&H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s)) = L^2(0,T;H^2_{\{0\}}(0,\ell_s))\cap H^1(0,T;L^2(0,\ell_s)),\\
&H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)) = L^2(0,T;H^4_{\{0,\ell_s\}}(0,\ell_s))\cap H^2(0,T;L^2(0,\ell_s)).
\end{aligned}$$

Tous les espaces précédents sont munis de leurs normes naturelles.

Nous introduisons l'espace des conditions d'entrée

$$\mathbf{H}(\Gamma_i) = \Big\{\mathbf{g}=(g_1,g_2) \mid g_2=0 \ \text{ et } \ g_1\in H^{\frac{3}{2}}(\Gamma_i)\cap H^1_0(\Gamma_i)\Big\}, \tag{1.3.17}$$

muni de la norme $(g_1,g_2)\mapsto \|g_1\|_{H^{\frac{3}{2}}(\Gamma_i)}$. Si nous identifions $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ avec son dual et si $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ désigne le dual de $\mathbf{V}^1_{\Gamma_d}(\Omega)$, nous avons

$$\mathbf{V}^1_{\Gamma_d}(\Omega)\hookrightarrow\mathbf{V}^0_{n,\Gamma_d}(\Omega)\hookrightarrow\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$$

avec des injections continues à image dense. Étant donnée $\beta>0$, nous introduisons les espaces de Sobolev avec poids

$$\begin{aligned}
\|\mathbf{w}\|_{\mathbf{H}^2_\beta} &:= \left(\sum_{|k|=0}^{2}\sum_{i=1}^{2}\int_\Omega\Big(\prod_{J\in\mathcal{J}} r_J^{2\beta}\Big)|\partial^k w_i|^2dx\right)^{1/2}, \quad \mathbf{w}\in C^\infty(\overline{\Omega};\mathbb{R}^2)\\
\|p\|_{H^1_\beta} &:= \left(\sum_{|k|=0}^{1}\int_\Omega\Big(\prod_{J\in\mathcal{J}} r_J^{2\beta}\Big)|\partial^k p|^2dx\right)^{1/2}, \quad p\in C^\infty(\overline{\Omega};\mathbb{R})
\end{aligned} \tag{1.3.18}$$

où r_J designe la distance au point de junction $J\in\mathcal{J}$, $k=(k_1,k_2)\in\mathbb{N}^2$ est un muti-indice de longueur $|k|=k_1+k_2$, ∂^k désigne l'opérateur différentiel partiel correspondant et $\mathbf{w}=(w_1,w_2)$. Nous désignons par $\mathbf{H}^2_\beta(\Omega;\mathbb{R}^2)$ (respectivement $H^1_\beta(\Omega)$) la fermeture de $C^\infty(\overline{\Omega};\mathbb{R}^2)$ (respectivement $C^\infty(\overline{\Omega})$) dans la norme $\|\cdot\|_{\mathbf{H}^2_\beta}$ (respectivement $\|\cdot\|_{H^1_\beta}$).

Soit $\varepsilon > 0$. Nous introduisons la fonction de troncature $\Psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfaisant $0 \le \Psi \le 1$,

$$\Psi = 1 \text{ on } (-L/2, \ell_s + \varepsilon/2) \times (-\ell, \ell) \text{ and } \Psi = 0 \text{ on } (\ell_s + \varepsilon, L) \times (-\ell, \ell). \tag{1.3.19}$$

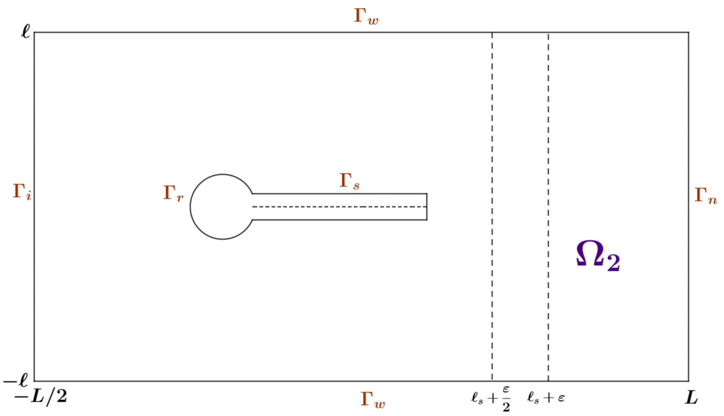


FIGURE 1.3 – Décomposition du domaine Ω.

Nous définissons également $\Omega_2 = (\ell_s + \varepsilon/2, L) \times (-\ell, \ell)$ (voir Figure 1.3) et introduisons les espaces de Sobolev hétérogènes

$$\begin{aligned}
\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) &= \left\{ \mathbf{F} \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1(\Omega) \right\}, \\
H_\delta^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1_\delta(\Omega) \right\}, \\
\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) &= \left\{ \mathbf{v} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{v} \in \mathbf{H}^2_\delta(\Omega) \right\},
\end{aligned} \tag{1.3.20}$$

munis de leurs normes naturelles.

Une difficulté propre aux problèmes d'interactions fluide-structure est le fait que le domaine du fluide change au cours du temps. Une manière de faire face à cette difficulté, et qui correspond à la stratégie que nous adopterons dans ce chapitre, consiste à définir un difféomorphisme X entre le domaine fixe Ω en fonction de η, que nous appellerons *domaine de référence*, et le domaine du fluide $\Omega_{\eta(t)}$ à l'instant t, que nous appellerons *domaine physique*. Cette transformation, qui est définie dans le Chapitre 3 (voir (3.2.9)), est utilisée pour définir le changement de variables suivant :

$$\widehat{\mathbf{u}}(t,z) = \mathbf{u}(t, X(t,z)) \ \text{ et } \ \widehat{p}(t,z) = p(t, X(t,z)), \tag{1.3.21}$$

pour tout $(t,z) \in (0,T) \times \Omega$. Ainsi, nous obtenons que $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ satisfait le système

$$\partial_t \widehat{\mathbf{u}} - \operatorname{div} \sigma(\widehat{\mathbf{u}}, \widehat{p}) = \widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}, \widehat{p}, \eta) \ \text{ dans } Q^T, \tag{1.3.22a}$$
$$\operatorname{div} \widehat{\mathbf{u}} = \operatorname{div} \widehat{\mathbf{G}}_{\text{div}}(\widehat{\mathbf{u}}, \eta) \ \text{ dans } Q^T, \tag{1.3.22b}$$
$$\widehat{\mathbf{u}} = \mathbf{g}^i \ \text{ sur } \Sigma_i^T, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_w^T, \ \ \widehat{\mathbf{u}} = 0 \ \text{ sur } \Sigma_r^T, \tag{1.3.22c}$$
$$\widehat{\mathbf{u}} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^T, \ \ \sigma(\widehat{\mathbf{u}}, \widehat{p})\mathbf{n} = 0 \ \text{ sur } \Sigma_n^T, \tag{1.3.22d}$$
$$\widehat{\mathbf{u}}(0) = \mathbf{u}_0 \ \text{ dans } \Omega, \tag{1.3.22e}$$
$$\partial_t^2 \eta + \alpha \Delta_s^2 \eta + \gamma B \eta_t = -\gamma_s^+ \widehat{p} + \gamma_s^- \widehat{p} + \widehat{F}_s(\widehat{\mathbf{u}}, \widehat{p}, \eta) \ \text{ dans } (0,T) \times (0, \ell_s), \tag{1.3.22f}$$
$$\eta = 0 \ \text{ et } \ \partial_{z_1} \eta = 0 \ \text{ sur } (0,T) \times \{0\}, \tag{1.3.22g}$$

$$\partial_{z_1}^2 \eta = 0 \text{ et } \partial_{z_1}^3 \eta = 0 \text{ sur } (0,T) \times \{\ell_s\}, \tag{1.3.22h}$$

$$\eta(0) = 0 \text{ et } \eta_t(0) = \eta_2^0 \text{ dans } (0,\ell_s), \tag{1.3.22i}$$

où $B = (\Delta_s^2)^{\frac{1}{2}}$, avec $\mathcal{D}(B) = H^2_{\{0\}}(0,\ell_s)$. Ici, les termes $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ et $\widehat{F}_s$ encapsulent les termes non linéaires obtenus après le changement de variables. Pour les epressions précises de ces termes, le lecteur est renvoyé au Chapitre 3 (voir (3.2.14), (3.2.15) et (3.2.16)).

De cette manière, nous avons réussi à réduire le système (1.3.16), formulé initialement sur le domaine physique $\Omega_{\eta(t)}$, à un système posé sur le domaine de référence Ω. Cependant, le prix à payer est l'apparition de nouvelles non-linéarités $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ et $\widehat{F}_s$ dans le système résultant (1.3.22). Comme nous verrons plus tard, ce sont précisément ces termes, ainsi que le fait de considérer des conditions aux limites libres sur une extrémité de la structure (voir (1.3.22h)), les principales difficultés rencontrées dans l'analyse du problème.

Avant de definir les espaces de Sobolev dans le domaine dépendant du temps $\Omega_{\eta(t)}$, nous rappelons que la définition précise de l'ensemble $E(0,T) \subset H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$ est introduite au Chapitre 3 (voir (3.2.8)). De même, les définitions des ensembles $\mathcal{O}_L$ et $\mathcal{O}_R$ sont données au Chapitre 3 (voir (3.2.10)).

Definition 1.3.1. *Soit $T > 0$. Pour tout $\eta \in E(0,T)$, nour dirons que $\mathbf{u}$ appartient à $L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,1}(\Omega_{\eta(\cdot)}))$ (respectivement à $H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega_{\eta(\cdot)}))$ s'il existe $X \in L^2(0,T;\mathbf{H}^{2+a_0}(\mathcal{O}_L)) \cap L^2(0,T;\mathbf{H}^4(\mathcal{O}_R)) \cap H^2(0,T;\mathbf{L}^2(\Omega))$, avec $0 < a_0 < 1/2$, tel que pour tout $t \in [0,T]$, $X(t,\cdot)$ est un $\mathcal{C}^1-$difféomorphisme de Ω sur $\Omega_{\eta(t)}$, et lorsque $\widehat{\mathbf{u}}$ définie par*

$$\widehat{\mathbf{u}}(t,z) = \mathbf{u}(t,X(t,z)), \quad \textit{pour tout } (t,z) \in [0,T]\times\Omega,$$

appartenant à $L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))$ (respectivement à $H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$). De même, nous dirons que p appartient à $L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega_{\eta(\cdot)}))$ lorsque $\widehat{p}$, défini par

$$\widehat{p}(t,z) = p(t,X(t,z)), \quad \textit{pour tout } (t,z) \in [0,T]\times\Omega,$$

appartient à $L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$.

Nous nous intéressons aux solutions $(\mathbf{u},p,\eta)$ du système (1.3.16) satisfaisant

$$\begin{aligned}
&\mathbf{u} \in L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega_{\eta(\cdot)})) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega_{\eta(\cdot)})),\\
&p \in L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega_{\eta(\cdot)})),\\
&\eta \in L^2(0,T;H^4_{\{0,\ell_s\}}(0,\ell_s)) \cap H^2(0,T;L^2(0,\ell_s)).
\end{aligned} \tag{1.3.23}$$

Definition 1.3.2. *Nous disons que le triplet $(\mathbf{u},p,\eta)$ est une solution forte du système (1.3.22) sur l'intervalle de temps $(0,T)$, lorsqu'il satisfait (1.3.23), les équations (1.3.16a)-(1.3.16b) au sens des distributions dans Q_η^T, l'équation (1.3.16f) au sens des distributions dans $(0,T)\times(0,\ell_s)$, les équations (1.3.16c)-(1.3.16d)-(1.3.16g)-(1.3.16h) au sens des traces, et les conditions initiales énoncées dans (1.3.16e) et (1.3.16i).*

Nous introduisons l'espace

$$\begin{aligned}
\mathcal{Z}_T = \Big(L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))\Big)&\\
\times L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)),&
\end{aligned} \tag{1.3.24}$$

muni de la norme

$$\begin{aligned}\|(\mathbf{u},p,\eta)\|_{\mathcal{Z}_T} = \|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ + \|p\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega))} + \|\eta\|_{H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))}.\end{aligned} \tag{1.3.25}$$

Nous introduisons également l'ensemble

$$\begin{aligned}B(T,R,\mathbf{u}_0,\eta_2^0) := \Big\{(\widehat{\mathbf{u}},\widehat{p},\eta) \in \mathcal{Z}_T \mid \|(\widehat{\mathbf{u}},\widehat{p},\eta)\|_{\mathcal{Z}_T} \le R, \ \ \eta \in E(0,T) \\ \text{et } \widehat{\mathbf{u}}(0) = \mathbf{u}_0, \ \eta(0) = 0, \ \eta_t(0) = \eta_2^0\Big\}.\end{aligned} \tag{1.3.26}$$

Le résultat principal du chapitre est le suivant :

Theorem 1.3.4. *Pour tout $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H^1_{\{0\}}(0,\ell_s)$ et $\mathbf{g}^i \in H^1_{\{0\}}(0,1;\mathbf{H}(\Gamma_i))$ satisfaisant*

$$\begin{aligned}&\mathbf{u}_0 = 0 \ \ \textit{sur } \Gamma_i, \ \ \mathbf{u}_0 = 0 \ \ \textit{sur } \Gamma_r \cup \Gamma_w, \\ &\mathbf{u}_0 = \eta_2^0(0,\cdot)\vec{\mathbf{e}}_2 \ \ \textit{sur } \Gamma_s, \ \ \operatorname{div}\mathbf{u}_0 = 0 \ \ \textit{dans } \Omega,\end{aligned} \tag{1.3.27}$$

il existe $T \in (0,1)$ et $R > 0$ tels que le système (1.3.22) admet une solution unique $(\widehat{\mathbf{u}},\widehat{p},\eta)$ dans $B(T,R,\mathbf{u}_0,\eta_2^0)$. De plus, si l'on pose

$$\mathbf{u}(t,x) = \widehat{\mathbf{u}}(t,X^{-1}(t,x)) \ \ \textit{et} \ \ p(t,x) = \widehat{p}(t,X^{-1}(t,x)), \ \ \textit{pour tout } x \in \Omega_{\eta(t)}, \ t \in [0,T],$$

où la transformation $X(t,\cdot) : \Omega \longrightarrow \Omega_{\eta(t)}$ est celle définie dans (3.2.9), alors $(\mathbf{u},p,\eta)$ est une solution du système (1.3.16).

Avant de décrire plus en détail les éléments qui font partie de la preuve, commençons par présenter l'idée générale de la stratégie.

1) *Description générale de la preuve.*

Comme mentionné ci-dessus, les principales difficultés résident dans la présence des termes non linéaires $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ et $\widehat{F}_s$ dans le système (1.3.22), ainsi que le fait de considérer des conditions aux limites libres sur une extrémité de la structure (voir (1.3.22h)). Nous expliquons ci-dessous ces point clés.

Pour traiter le problème non linéaire (1.3.22), nous utilisons une approche classique, qui a été utilisée par exemple dans [MRR20] et [FNR19]. Tout d'abord, nous associons au problème non linéaire un problème linéaire (PL) avec des termes sources non homogènes, ces derniers représentant les non linéarités. Ensuite, après avoir résolu le problème linéaire, nous appliquons le Théorème du point fixe de Banach pour traiter le problème non linéaire. Dans cette analyse, les points cruciaux sont l'établissement du caractère bien posé du problème linéaire (PL) et les estimations de type Lipschitz utilisées dans l'argument de point fixe. En effet, ces deux étapes sont interdépendantes dans le sens où la régularité imposée aux données non homogènes du problème linéaire (PL) doit être appropriée pour obtenir les estimations des termes non linéaires dans l'argument de point fixe. Cet aspect central se ramène à l'étude de la régularité spatiale de la vitesse et de la pression du fluide dans le problème linéaire non homogène. Plus précisément, nous devons étudier la régularité de la vitesse $\mathbf{w}$ et la pression π du système de Stokes stationnaire

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{w},\pi) = \mathbf{F} \ \ \text{dans } \Omega, \ \ \operatorname{div}\mathbf{w} = h \ \ \text{dans } \Omega, \\ \mathbf{w} = \mathbf{g} \ \ \text{sur } \Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n} = 0 \ \ \text{sur } \Gamma_n, \end{cases} \tag{1.3.28}$$

où $\mathbf{F}$, h et $\mathbf{g}$ sont données et jouent le rôle de termes non linéaires. Il y a deux difficultés sousjacents dans l'étude de ce problème. Tout d'abord, le domaine Ω possède des coins rentrants aux points A et B (voir Figure 1.2). Deuxièmement, nous remarquons aussi la présence des points de junction entre les conditions aux limites de Dirichlet et de Neumann aux sommets C et D. En suivant la démarche de [FNR19] (voir aussi [MRR20]), il devrait être suffisant d'utiliser le résultat de régularité (R1) déjà mentionné dans la Section 1.3.1, qui affirme que

$$\begin{aligned}&(\mathbf{w},\pi) \text{ appartient à l'espace de Sobolev pondéré } \mathbf{H}^2_\delta(\Omega)\times H^1_\delta(\Omega)\\ &\text{lorsque } \mathbf{F}\in\mathbf{L}^2_\delta(\Omega),\ h\in H^1_\delta(\Omega) \text{ et } \mathbf{g}\in\mathbf{H}^{\frac{3}{2}}_\delta(\Gamma_d),\end{aligned}\tag{R1}$$

pour tout $\delta\in(\delta^*,1)$, où $\delta^*\in(0,1/2)$. Cependant, du moins dans la manière dont les auteurs de [FNR19] utilisent le résultat de régularité (R1), il n'est pas possible de le reproduire dans notre cas. Examinons cela plus en détail. Les auteurs de [FNR19] utilisent le résultat de régularité (R1) dans le cas où $\mathbf{F}\in\mathbf{L}^2(\Omega)$, $h\in H^1(\Omega)$ et $\mathbf{g}\in\mathbf{H}^{\frac{3}{2}}(\Gamma_i)$. Le fait de considérer $\mathbf{F}\in\mathbf{L}^2(\Omega)$ par les auteurs de [FNR19] est suffisant pour estimer tous les termes non linéaires encapsulés dans $\mathbf{F}$ dans la norme $\mathbf{L}^2(\Omega)$. En particulier, grâce à [FNR19, Lemme 6.3], qui établit que si $\mathbf{w}\in\mathbf{H}^2_\delta(\Omega)$ et $\eta\in H^1(-L/2,L)$ avec $\eta(0)=\eta(\ell_s)=0$, alors $\mathbf{w}_{xx}\eta$ appartient à $\mathbf{L}^2(\Omega)$. Cependant, dans notre cas nous sommes incapables d'estimer ce terme. La raison en est que le cœur de la démonstration du [FNR19, Lemme 6.3] repose sur le fait que $\eta(0)=\eta(\ell_s)=0$, ce qui n'est pas vérifié dans notre situation, puisque en général, $\eta(\ell_s)\neq 0$ (voir (1.3.22h)). C'est précisément à cause de cet argument que les conditions aux limites libres sur une extrémité de la structure posent des difficultés. Pour cette raisson, nous devons chercher un résultat de régularité alternatif. En particulier, une version adaptée de (R1) est présentée dans le Théorème 2.3.2 du Chapitre 2, mais en termes d'espaces de Sobolev hétérogènes.

Une option alternative à celle présentée et qui n'a pas été explorée dans ce travail de thèse, consiste à travailler directement dans les espaces avec poids. Cela impliquerait, comme point de départ, d'analyser une décomposition de Leray de l'espace avec poid $\mathbf{L}^2_\delta(\Omega)$, ce qui revient à étudier un système elliptique avec des conditions aux limites mixtes dans un domaine avec des coins rentrants.

Le preuve du Théorème 1.3.4 repose essentiellement sur deux arguments :

2) *Étude du système linéaire non homogène.*

Nous considérons le système fluide-structure linearisé

$$\begin{cases}\partial_t\mathbf{v}-\operatorname{div}\sigma(\mathbf{v},p)=\mathbf{F}_f \text{ dans } Q^T,\\ \operatorname{div}\mathbf{v}=\operatorname{div}\mathbf{G}_{\mathrm{div}} \text{ dans } Q^T,\\ \mathbf{v}=\mathbf{g}^i \text{ sur } \Sigma^T_i,\ \ \mathbf{v}=0 \text{ sur } \Sigma^T_w\cup\Sigma^T_r,\ \ \mathbf{v}=\zeta_2\vec{\mathbf{e}}_2 \text{ sur } \Sigma^T_s,\\ \sigma(\mathbf{v},p)\mathbf{n}=0 \text{ sur } \Sigma^T_n,\\ \mathbf{v}(0)=\mathbf{v}_0 \text{ dans } \Omega,\\ \partial_t\zeta_1=\zeta_2 \text{ sur } (0,T)\times(0,\ell_s),\\ \partial_t\zeta_2+\alpha\Delta_s^2\zeta_1+\gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2=-\gamma_s^+p+\gamma_s^-p+F_s \text{ dans } (0,T)\times(0,\ell_s),\\ \zeta_1=0 \text{ et } \partial_{x_1}\zeta_1=0 \text{ sur } (0,T)\times\{0\},\\ \partial^2_{x_1}\zeta_1=0 \text{ et } \partial^3_{x_1}\zeta_1=0 \text{ sur } (0,T)\times\{\ell_s\},\\ \zeta_1(0)=0 \text{ et } \zeta_2(0)=\zeta_2^0 \text{ dans } (0,\ell_s).\end{cases}\tag{1.3.29}$$

Nous prouvons le théorème suivant :

Theorem 1.3.5. *Soit* $0<T<1$. *Supposons que* $\mathbf{v}_0\in\mathbf{H}^1(\Omega)$, $\zeta_2^0\in H^1_{\{0\}}(0,\ell_s)$, $\mathbf{F}_f\in L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$, $\mathbf{g}^i\in H^1_{\{0\}}(0,1;\mathbf{H}(\Gamma_i))$ *et* $F_s\in L^2(0,T;L^2(0,\ell_s))$. *Supposons également* *que* $\mathbf{G}_{\mathrm{div}}$ *satisfait les conditions énoncées en* (3.4.9), (3.4.10), (3.4.11) *et* $\mathbf{G}_{\mathrm{div}}|_{t=0} = 0$. *De plus, supposons que les conditions de compatibilité* (3.4.14) *sont satisfaites. Alors, le système* (1.3.29) *admet une solution unique* $(\mathbf{v}, p, \zeta_1, \zeta_2)$ *appartenant à* $L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \times L^2(0,T;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)) \times H^{4,2}((0,T)\times(0,\ell_s)) \times H^{2,1}((0,T)\times(0,\ell_s))$.

Idée de la preuve. Les trois points essentiels de la démonstration sont les suivants :

(a) *Constrution d'un relèvement pour* $\mathbf{g}^i$ *et* $\mathbf{G}_{\mathrm{div}}$.

La construction du relèvement $\mathbf{z}$ pour $\mathbf{g}^i$ est une conséquence du résultat de régularité donné dans le Théorème 3.3.1(ii) (voir Proposition 3.4.1). D'autre part, la construction du relèvement $\mathbf{w}$ associé à $\mathbf{G}_{\mathrm{div}}$ repose sur une combinaison des approches utilisées dans [FNR19, Theorem 10.2] et [MRR20, Proposition 5.1] (voir Proposition 3.4.2).

(b) *Réécriture du système* (1.3.29) *avec* $\mathbf{g}^i = 0$ *et* $\mathbf{G}_{\mathrm{div}} = 0$ *en termes d'opérateurs.*

Une fois introduits les relèvements $\mathbf{z}$ et $\mathbf{w}$, nous utilisons le Théorème 3.3.6 pour réécrire le système (1.3.29), avec $\mathbf{g}^i = 0$ and $\mathbf{G}_{\mathrm{div}} = 0$, sous la forme d'une équation opératorielle.

(c) *Régularité de la solution du système* (1.3.29) *avec* $\mathbf{g}^i = 0$ *et* $\mathbf{G}_{\mathrm{div}} = 0$.

Tout d'abord, nous utilisons l'analyticité du semi-groupe associé à l'opérateur fluide-structure défini en (3.3.27) (voir Théorème 3.3.7), afin de déduire, grâce au résultat de régularité maximale [BDDM07, Theorem 3.1, p. 143], la régularité en temps. Ensuite, la régularité spatiale découle du Théorème 3.3.1(ii).

3) *Estimations des termes non linéaires et argument de point fixe.*

Après avoir établi le caractère bien posé du système (1.3.29), la dernière étape de la démonstration du Théorème 1.3.4 consiste à estimer les termes non linéaires et à utiliser un argument de point fixe. Pour cela, nous considérons l'application $\mathcal{N} : B(T,R,\mathbf{u}_0,\eta_2^0) \longrightarrow B(T,R,\mathbf{u}_0,\eta_2^0)$ définie par

$$\mathcal{N}(\mathbf{\Phi},\psi,k) = (\widehat{\mathbf{u}},\widehat{p},\eta) \quad \text{pour tout } (\mathbf{\Phi},\psi,k) \in B(T,R,\mathbf{u}_0,\eta_2^0),$$

où $(\widehat{\mathbf{u}},\widehat{p},\eta)$ est solution du système

$$\begin{cases} \partial_t \widehat{\mathbf{u}} - \operatorname{div}\sigma(\widehat{\mathbf{u}},\widehat{p}) = \widehat{\mathbf{F}}_f(\mathbf{\Phi},\psi,k), \ \ \operatorname{div}\widehat{\mathbf{u}} = \operatorname{div}\widehat{\mathbf{G}}_{\mathrm{div}}(\mathbf{\Phi},k) \ \text{ dans } Q^T, \\ \widehat{\mathbf{u}} = \mathbf{g}^i \ \text{ sur } \Sigma_i^T, \ \ \widehat{\mathbf{u}} = 0 \ \text{ sur } \Sigma_w^T \cup \Sigma_r^T, \ \ \widehat{\mathbf{u}} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^T, \\ \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \ \text{ sur } \Sigma_n^T, \ \ \widehat{\mathbf{u}}(0) = \mathbf{u}_0(X_\eta(\cdot)) \ \text{ dans } \Omega, \\ \partial_t^2 \eta_t + \alpha\Delta_s^2\eta + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_t = -\gamma_s^+\widehat{p} + \gamma_s^-\widehat{p} + \widehat{F}_s(\mathbf{\Phi},k) \ \text{ dans } (0,T)\times(0,\ell_s), \\ \eta = 0 \ \text{ et } \ \partial_{x_1}\eta = 0 \ \text{ sur } (0,T)\times\{0\}, \\ \partial_{x_1}^2\eta = 0 \ \text{ et } \ \partial_{x_1}^3\eta = 0 \ \text{ sur } (0,T)\times\{\ell_s\}, \\ \eta(0) = 0 \ \text{ et } \ \eta_t(0) = \eta_2^0 \ \text{ dans } (0,\ell_s). \end{cases} \tag{1.3.30}$$

Pour montrer que l'application $\mathcal{N}$ admet un unique point fixe, nous utilisons le Théorème du point fixe de Banach. On peut en fait montrer que si T est suffisamment petit, l'application $\mathcal{N}$ est bien définie, et qu'elle est une contraction stricte. Ces derniers points découlent des estimations précises des termes non linéaires $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ et $\widehat{F}_s$ (voir Lemmes 3.5.5, 3.5.6 et 3.5.7).

1.3.3 Chapitre 4 : Simulations numériques du système d'interaction fluide-structure

Tout au long de cette partie, nous considérons la configuration geométrique représentée dans la Figure 1.2, dont la description précise a été introduite dans la Section 1.2 (voir (1.2.1), (1.2.2) et (1.2.4)).

- **Formulation du problème et littérature existante**

Dans ce chapitre, nous nous intéressons à l'approximation numérique de la solution du système

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ dans } Q^T_\eta, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ dans } Q^T_\eta, \\ \mathbf{u} = \mathbf{g}^i \ \text{ sur } \Sigma^T_i, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma^T_r \cup \Sigma^T_w, \\ \mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma^T_\eta, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ sur } \Sigma^T_n, \\ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ dans } \Omega, \\ \partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) + f_s \text{ dans } (0,T)\times(0,\ell_s), \\ \eta = 0 \ \text{ et } \ \partial_{x_1}\eta = 0 \ \text{ sur } (0,T)\times\{0\}, \\ \partial^2_{x_1}\eta = 0 \ \text{ et } \ \partial^3_{x_1}\eta = 0 \ \text{ sur } (0,T)\times\{\ell_s\}, \\ \eta(0) = 0 \ \text{ et } \ \partial_t\eta(0) = \eta^0_2 \ \text{ dans } (0,\ell_s), \end{cases} \tag{1.3.31}$$

où $\mathbf{u}$ et p représentent respectivement la vitesse et la pression du fluide. Ici, $\sigma(\mathbf{u},p)$ est le tenseur de contraintes du fluide donné par

$$\sigma(\mathbf{u},p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

avec $\nu > 0$ représentant la viscosité du fluide. La condition au bord d'entrée $\mathbf{g}^i = \mathbf{g}^i_s + \beta\mathbf{g}^i_p$, où $\mathbf{g}^i_s$ est indépendante du temps, $\mathbf{g}^i_p$ est une perturbation dépendant du temps, tandis que β représente l'amplitude de la perturbation.

L'opérateur d'amortissement B est donné par :

$$B = \Delta_s^2 = \partial^4_{x_1}, \ \ D(B) = H^4_{\{0,\ell_s\}}(0,\ell_s).$$

L'expression de la force H est donnée par

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2. \tag{1.3.32}$$

Comme dans l'étude du problème d'existence de solutions pour le système (1.3.31), une première difficulté inhérente à ce type de système est le fait que le domaine du fluide change au cours du temps. Cette problématique est également un obstacle lorsqu'on s'intéresse à l'approximation numérique du système. Pour faire face à cette difficulté, nous utilisons la méthode ALE (Arbitrary Lagrangian Eulerian) (voir, par exemple, [DGH82], [QTV00], [QF04], [TH06], [FGG07], [Ric15]).

En ce qui concerne la résolution numérique du système (1.3.31), deux grande stratégies sont utilisées dans la littérature. La première, connue sous le nom de *partitionnée*, consiste à résoudre séparément les sous-problèmes du fluide et de la structure, qui sont ensuite couplés à travers des conditions de transmission. Cette stratégie présente l'avantage de permettre l'utilisation de solveurs ad hoc déjà existants. Cependant, cela se fait au prix d'une perte d'efficacité par rapport au second groupe décrit ci-dessous. Ce second groupe, appelé approche *monolithique*, est caractérisé par le fait que les sous-problèmes fluide et structure sont résolus simultanément. Comme souligné dans [Ric15], "cette approche permet l'utilisation de techniques de discrétisation implicites et de solveurs fortement couplés pour l'ensemble du système". L'un des principaux inconvénient de cette stratégie réside dans son coût de calcul élevé. La stratégie utilisée dans ce chapitre suit l'approche monolithique présentée dans [Mur19]. Plus précisément, un algorithme semi-implicite est employé, où le mot *semi-implicite* est compris dans le sens où le domaine fluide est calculé explicitement.

L'approche suivie dans ce chapitre, la méthode ALE et l'algorithme semi-implicite 1, est également la stratégie utilisée dans les simulations numériques du problème de stabilisation traité au Chapitre 6.

• **Présentation de nos résultats**

Nous considérons d'abord la transformation ALE $\mathcal{A}(t,\cdot) : \Omega_{ref} \longrightarrow \Omega_{\eta(t)}$ définie par

$$\mathcal{A}(t,\cdot) = I + \int_0^t \mathbf{w}(s,\cdot)\,ds, \tag{1.3.33}$$

où $\mathbf{w}(t,\cdot)$ est solution de l'équation elliptique

$$\Delta\mathbf{w} = 0 \ \text{ dans } \Omega, \ \ \mathbf{w} = \mathbf{u}|_{\Gamma_s} \ \text{ sur } \Gamma_s, \ \ \mathbf{w} = 0 \ \text{ sur } \Gamma \setminus \Gamma_s. \tag{1.3.34}$$

Ici, $\mathbf{u}|_{\Gamma_s}$ représente la trace de la vitesse du fluide sur Γ_s et $\mathbf{w}$ un relèvement.

La discrétisation en temps est traitée à l'aide de la méthode classique d'Euler implicite. Nous désignons par Δt le pas de temps et $t^k = k\Delta t$, pour $k \in \mathbb{N}$ le niveau de temps k. Pour tout $k \in \mathbb{N}$, $\Omega^k := \mathcal{A}(t^k, \Omega_{ref})$ avec frontière $\Gamma^k = \Gamma_i \cup \Gamma_0 \cup \Gamma_s^k \cup \Gamma_n$, où $\Gamma_0 = \Gamma_r \cup \Gamma_w$ et $\Gamma_s^k = \Gamma_{s,top}^k \cup \Gamma_{s,bot}^k \cup \Gamma_{s,lat}^k$ (voir Figure 1.4). Nous désignons par $\mathbf{u}^k$, p^k et $\boldsymbol{\lambda}^k$ les approximations de $\mathbf{u}(t^k,\cdot)$, $p(t^k,\cdot)$ et $\boldsymbol{\lambda}(t^k,\cdot)$, respectivement. Ici, $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}_i^k, \boldsymbol{\lambda}_0^k, \boldsymbol{\lambda}_{s,top}^k, \boldsymbol{\lambda}_{s,bot}^k, \boldsymbol{\lambda}_{s,lat}^k)^\top$ représentent les multiplicateurs de Langrange associés aux conditions de bord Dirichlet. Nous notons également η_1^k et η_2^k les approximations de $\eta_1(t^k,\cdot)$ et $\eta_2(t^k,\cdot)$ définies sur $(0,\ell_s)$, respectivement.

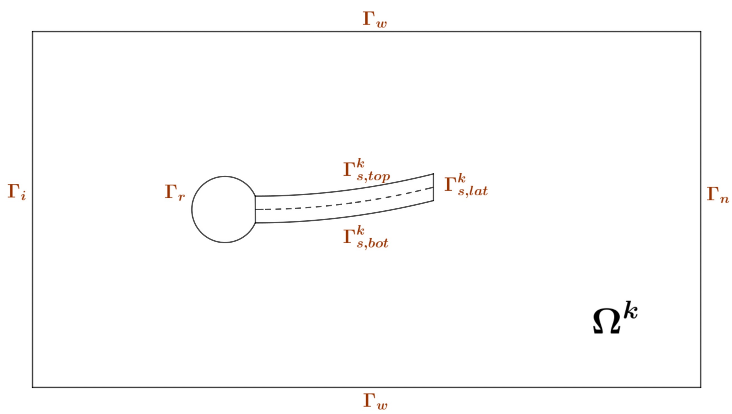


Figure 1.4 – Domaine physique au niveau de temps k.

Avant d'introduire l'algorithme de résolution, nous considérons le problème intermédiaire suivant. Étant donné $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k et η_2^k :

Trouver $\widehat{\mathbf{u}}^{k+1} \in \mathbf{H}^1(\Omega^k)$, $\widehat{p}^{k+1} \in L^2(\Omega^k)$, $\widehat{\boldsymbol{\lambda}}^{k+1} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k \setminus \Gamma_n)$, $\eta_1^{k+1}, \eta_2^{k+1} \in H^2_{\{0\}}(0,\ell_s)$ tels que

$$
\left\{
\begin{aligned}
&\int_{\Omega^k} \frac{\widehat{\mathbf{u}}^{k+1} - \mathbf{u}^k}{\Delta t} \cdot \boldsymbol{\phi} = a_f(\widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) + b(\boldsymbol{\phi}, \widehat{p}^{k+1}) + c(\widehat{\mathbf{u}}^{k+1} - \mathbf{w}^k, \widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) \\
&\qquad\qquad + \int_{\Gamma_i} \widehat{\boldsymbol{\lambda}}_i^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma_0^k} \widehat{\boldsymbol{\lambda}}_0^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma^k_{s,top}} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \boldsymbol{\phi} \\
&\qquad\qquad + \int_{\Gamma^k_{s,bot}} \widehat{\boldsymbol{\lambda}}_{s,bot}^{k+1} \cdot \boldsymbol{\phi}, + \int_{\Gamma^k_{s,lat}} \widehat{\boldsymbol{\lambda}}_{s,lat}^{k+1} \cdot \boldsymbol{\phi}, \quad \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega^k), \\
&b(\widehat{\mathbf{u}}^{k+1}, \psi) = 0, \quad \forall \psi \in L^2(\Omega^k), \\
&\int_{\Gamma_i} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_i} \mathbf{g}^i \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i), \qquad \int_{\Gamma_0} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_0), \\
&\int_{\Gamma^k_{s,top}} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma^k_{s,top}} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k_{s,top}), \\
&\int_{\Gamma^k_{s,bot}} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma^k_{s,bot}} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k_{s,bot}), \\
&\int_{\Gamma^k_{s,lat}} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma^k_{s,lat}} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k_{s,lat}), \\
&\int_0^{\ell_s} \frac{\eta_1^{k+1} - \eta_1^k}{\Delta t} \zeta = \int_0^{\ell_s} \eta_2^{k+1} \zeta, \quad \forall \zeta \in H^2_{\{0\}}(0,\ell_s), \\
&\int_0^{\ell_s} \frac{\eta_2^{k+1} - \eta_2^k}{\Delta t} \zeta = a_s^1(\eta_1^{k+1}, \zeta) + a_s^2(\eta_2^{k+1}, \zeta) \\
&\qquad\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} \\
&\qquad\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{bot}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} + \int_0^{\ell_s} f_s \zeta, \quad \forall \zeta \in H^2_{\{0\}}(0,\ell_s).
\end{aligned}
\right.
\tag{1.3.35}
$$

où

$$
a_f(\mathbf{v}, \boldsymbol{\phi}) = -2\nu \int_{\Omega^k} \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\phi}), \quad b(\boldsymbol{\phi}, q) = \int_{\Omega^k} (\operatorname{div} \boldsymbol{\phi})\, q, \quad c(\mathbf{v}, \mathbf{v}, \boldsymbol{\phi}) = \int_{\Omega^k} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\phi},
$$
$$
a_s^1(\eta_1, \zeta) = -\alpha \int_0^{\ell_s} \Delta\eta_1 \cdot \Delta\zeta, \quad a_s^2(\eta_2, \zeta) = -\gamma \int_0^{\ell_s} \Delta\eta_2 \cdot \Delta\zeta.
$$

Algorithm 1: Semi-implicit algorithm

Pour $k \geq 1$:

1 : Résoudre le système linéaire obtenu après l'application de l'algorithme de Newton au système (1.3.35) pour obtenir $\widehat{\mathbf{u}}^{k+1}$, $\widehat{p}^{k+1}$, $\widehat{\boldsymbol{\lambda}}^{k+1}$, η_1^{k+1}, η_2^{k+1}.

2 : Calculer la vitesse du maillage $\widehat{\mathbf{w}}^{k+1} : \Omega^k \to \mathbb{R}^2$ satisfaisant l'équation elliptique

$$
\left\{
\begin{aligned}
&\Delta \widehat{\mathbf{w}}^{k+1} = 0 \ \text{ dans } \Omega^k, \\
&\widehat{\mathbf{w}}^{k+1} = \widehat{\mathbf{u}}^{k+1} \ \text{ sur } \Gamma_s^k, \\
&\widehat{\mathbf{w}}^{k+1} = 0 \ \text{ sur } \Gamma \setminus \Gamma_s^k.
\end{aligned}
\right.
\tag{1.3.36}
$$

3 : Définir $\mathcal{A}^k(\widehat{\mathbf{x}}) := \widehat{\mathbf{x}} + \Delta t \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}})$ et $\Omega^{k+1} := \mathcal{A}^k(\Omega^k)$.

4 : Définir $\mathbf{u}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$, $p : \Omega^{k+1} \to \mathbb{R}$, $\boldsymbol{\lambda}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ et $\mathbf{w}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ par

$$
\begin{aligned}
&\mathbf{u}^{k+1}(\mathbf{x}) = \widehat{\mathbf{u}}^{k+1}(\widehat{\mathbf{x}}), \ \ p^{k+1}(x) = \widehat{p}^{k+1}(\widehat{\mathbf{x}}), \ \ \boldsymbol{\lambda}^{k+1}(x) = \widehat{\boldsymbol{\lambda}}^{k+1}(\widehat{\mathbf{x}}) \\
&\text{et } \ \mathbf{w}^{k+1}(\mathbf{x}) = \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}}), \ \ \forall \mathbf{x} = \mathcal{A}^k(\widehat{\mathbf{x}}), \widehat{\mathbf{x}} \in \Omega^k.
\end{aligned}
$$

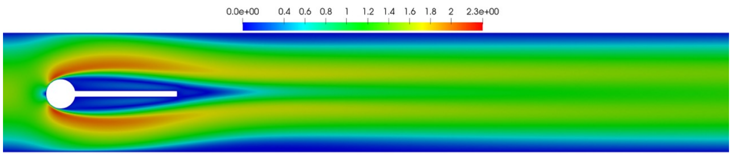


FIGURE 1.5 – Module de la vitesse $\mathbf{U}_s$ du fluide correspondant à $Re = 200$ (solution stationnaire).

1. **Analyse spectrale avec différents paramètres physiques**.

En ce qui concerne l'étude du problème de stabilisation (voir Chapitre 6), un élément important consiste à calculer numériquement le spectre du système linéarisé autour d'une solution stationnaire donnée (dans la Figure 1.5, nous montrons le module de la vitesse du fluide stationnaire pour un nombre de Reynolds $Re = 200$). Dans ce cadre, nous analysons le spectre d'un tel système en fixant le nombre de Reynolds $Re = 200$ ainsi que le coefficient d'amortissement $\gamma = 10^{-6}$ de la structure. Par exemple, dans la Figure 1.6, nous montrons une partie du spectre correspondante à un coefficient de rigidité $\alpha = 10^{-1}$. Dans cette figure, nous observons deux valeurs propres conjuguées instables, que nous désignons par $\mu_{1,2}$ et $\mu_{3,4}$, dont les valeurs précises sont présentées dans le Tableau 1.1

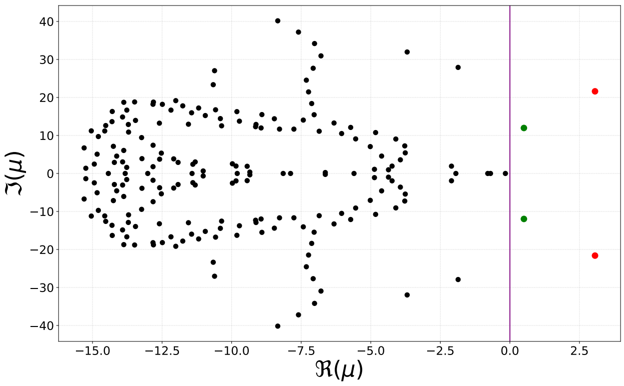


FIGURE 1.6 – Partie du spectre fluide-structure correspondant à $Re = 200$, un coefficient de rigidité $\boldsymbol{\alpha = 10^{-1}}$, et un coefficient d'amortissement $\gamma = 10^{-6}$.

$\boldsymbol{\mu_{1,2}}$	$\boldsymbol{\mu_{3,4}}$	$\boldsymbol{\mu_5}$	$\boldsymbol{\mu_6}$	$\boldsymbol{\mu_7}$	$\boldsymbol{\mu_{8,9}}$	$\boldsymbol{\mu_{10}}$
$3.05 \pm 21.61i$	$0.50 \pm 11.96i$	-0.16	-0.69	-0.79	$-1.86 \pm 27.91i$	-1.94

TABLEAU 1.1 – Premières valeurs propres du système fluide-structure (ordonnées selon la partie réelle) correspondant à un nombre de Reynolds $Re = 200$, coefficient de rigidité $\boldsymbol{\alpha = 10^{-1}}$, et coefficient d'amortissement $\gamma = 10^{-6}$.

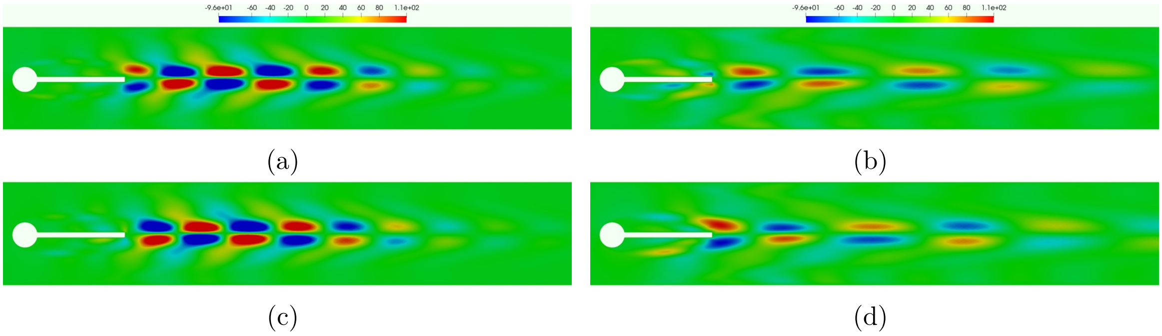


FIGURE 1.7 – Partie réelle de la composante horizontale de la vitesse du fluide associée aux valeurs propres instables $\mu_{1,2}$ et $\mu_{3,4}$, pour $\boldsymbol{\alpha = 10^{-1}}$. (a)-(c) Fonctions propres associées à μ_1 et μ_2, respectivement. (b)-(d) Fonctions propres associées à μ_3 et μ_4, respectivement.

2. **Simulations numériques du problème direct**.

En utilisant l'algorithme 1, nous présentons plusieurs simulations numériques dans le même esprit que pour l'analyse du spectre : nous fixons le nombre de Reynolds à $Re = 200$ et le coefficient d'amortissement $\gamma = 10^{-6}$. Par exemple, dans la Figure 1.8, nous montrons le module de la vitesse du fluide à différents instants, correspondant à une valeur de coefficient de rigidité $\alpha = 10^{-1}$. La Figure 1.9 montre le déplacement de la structure aux mêmes instants.

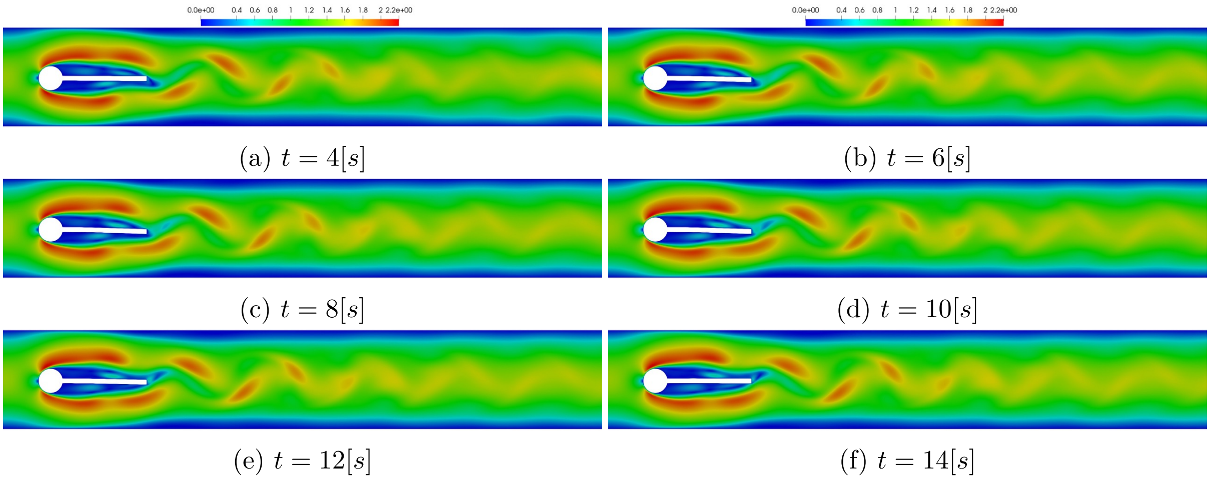
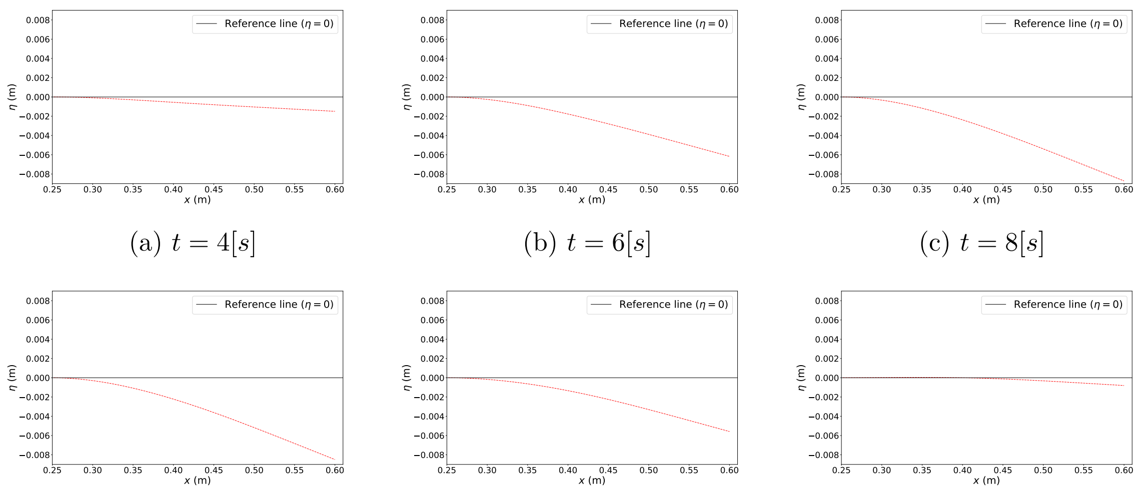


FIGURE 1.8 – Snapshots du module de la vitesse à différents instants correspondant au coefficient de rigidité $\alpha = 10^{-1}$ et au paramètre de perturbation $\boldsymbol{\beta = 1.5}$ pour $Re = 200$.



(a) $t = 4[s]$ (b) $t = 6[s]$ (c) $t = 8[s]$

(d) $t = 10[s]$ (e) $t = 12[s]$ (f) $t = 14[s]$

FIGURE 1.9 – Snapshots du déplacement de la structure (ligne rouge en pointillés) à différents instants correspondant au coefficient de rigidité $\alpha = 10^{-1}$ et au paramètre de perturbation $\boldsymbol{\beta = 1.5}$.

1.3.4 Chapitre 5 : Stabilisation du système d'interaction fluide-structure

• Formulation du problème et littérature existante

Commençons par établir la notation qui sera utilisée tout au long de cette section.

$$\begin{aligned}
&Q_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Omega_{\eta(t)}\Big),\ \Sigma_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Gamma_{\eta(t)}\Big),\\
&Q^\infty = (0,\infty)\times\Omega,\ \ \Sigma_s^\infty = (0,\infty)\times\Gamma_s,\\
&\Sigma_i^\infty = (0,\infty)\times\Gamma_i,\ \ \Sigma_d^\infty = (0,\infty)\times\Gamma_d,\\
&\Sigma_r^\infty = (0,\infty)\times\Gamma_r,\ \ \Sigma_n^\infty = (0,\infty)\times\Gamma_n.
\end{aligned} \tag{1.3.37}$$

Soit $(\mathbf{u}_s, p_s)$ solution du système

$$\begin{cases}
(\mathbf{u}_s\cdot\nabla)\mathbf{u}_s - \operatorname{div}\sigma(\mathbf{u}_s,p_s) = 0, \ \text{ dans } \Omega,\\
\operatorname{div}\mathbf{u}_s = 0 \ \text{ dans } \Omega,\\
\mathbf{u}_s = \mathbf{g}_s, \ \text{ sur } \Gamma_i, \ \ \mathbf{u}_s = 0 \ \text{ sur } \Gamma\setminus\Gamma_i,\\
\sigma(\mathbf{u}_s,p_s)\mathbf{n} = 0 \ \text{ sur } \Gamma_n.
\end{cases} \tag{1.3.38}$$

Considérons le système d'interaction fluide-structure

$$\left\{\begin{array}{l} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ dans } Q_\eta^\infty, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ dans } Q_\eta^\infty, \\ \mathbf{u} = \mathbf{g} \ \text{ sur } \Sigma_i^\infty, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma_w^\infty, \ \ \mathbf{u} = 0 \ \text{ sur } \Sigma_r^\infty, \\ \mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_\eta^\infty, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ sur } \Sigma_n^\infty, \\ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ dans } \Omega, \\ \partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_t = H(\mathbf{u},p,\eta) + f_s + f \text{ dans } (0,\infty)\times(0,\ell_s), \\ \eta = 0 \ \text{ et } \ \partial_{x_1}\eta = 0 \ \text{ sur } (0,\infty)\times\{0\}, \\ \partial_{x_1}^2\eta = 0 \ \text{ et } \ \partial_{x_1}^3\eta = 0 \ \text{ sur } (0,\infty)\times\{\ell_s\}, \\ \eta(0) = 0 \ \text{ et } \ \partial_t\eta(0) = \eta_2^0 \ \text{ dans } (0,\ell_s), \end{array}\right. \tag{1.3.39}$$

où

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2,$$

avec

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x_1,\eta(t,x_1)\pm e), p(t,x_1,\eta(t,x_1)\pm e)).$$

La condition au bord en entrée $\mathbf{g} = \mathbf{g}_s + \beta\mathbf{g}_p$, où $\mathbf{g}_s$ est indépendante du temps et $\mathbf{g}_p$ est une perturbation dépendant du temps. La fonction f_s est indépendante du temps et est choisie de telle sorte que le triplet $(\mathbf{u},\eta,\eta_t) = (\mathbf{u}_s,0,0)$ constitue une solution stationnaire du système (1.3.39). Ainsi,

$$f_s = \sigma(\mathbf{u}_s,p_s)\mathbf{n}^+\cdot\vec{\mathbf{e}}_2 + \sigma(\mathbf{u}_s,p_s)\mathbf{n}^-\cdot\vec{\mathbf{e}}_2, \tag{1.3.40}$$

où $\mathbf{n}^+$ (respectivement $\mathbf{n}^-$) représente la normale unitaire exterièure à Γ_s^+ (respectivement Γ_s^-).

Le but de ce chapitre est de trouver un contrôle sous la forme $f = \sum_{j=1}^{N_c} f_j(t)w_j(z_1)$, donné en boucle fermée, capable de stabiliser le système (1.3.39) avec un taux de décroissance exponentielle prescrit $\omega > 0$, localement autour $(\mathbf{u},\eta,\eta_t) = (\mathbf{u}_s,0,0)$. Ici, les profils w_j sont choisis de manière appropriée afin de garantir une propriété de stabilisabilité.

Dans ce qui suit, nous présentons une liste non exhaustive de travaux portant sur la stabilisation de modèles fluide-structure.

À notre connaissance, un premier travail dans cette direction est l'article de Raymond [Ray10], où il considère un système fluide-structure couplant les équations de Navier-Stokes incompressibles dans un domaine rectangulaire $2D$ avec des conditions de Dirichlet, et une équation d'Euler-Bernoulli amortie avec des conditions encastrées. Le but de ce travail est de trouver un contrôle en boucle fermée, capable de stabiliser expontiellement le système autour de la solution stationnaire nulle, avec un taux de décroissance arbitraire. Dans la même ligne, les auteurs de [FNR19], étudient un système similaire dans lequel, au lieu de considérer des conditions Dirichlet pour le fluide, ils ont considéré des conditions mixtes de type Dirichlet-Neumann. Dans cet article, les auteurs ont stabilisé le système autour d'une solution stationnaire qui n'est pas nécessairement nulle, avec un contrôle de dimension finie en boucle fermée. Dans le cadre du même modèle analysé dans [Ray10], mais dans le contexte de solutions faibles, les auteurs de [BT17] étudient le problème de stabilisation du système autour d'une solution stationnaire qui n'est pas nécessairement nulle, avec un contrôle donné sous la forme en boucle fermée qui agit sur le domaine externe du fluide. Dans [MR17], les auteurs étudient un système couplant les équations de Navier-Stokes incompressibles et une équation des ondes amortie. Ils prennent également en compte des conditions aux limites de type periódique. La stabilisation est effectuée autour de la solution stationnaire nulle, avec un contrôle de dimension finie en boucle fermée qui agit sur une partie de la frontière du fluide. Dans un contexte légèrement différent en termes de dynamique régissant la structure, dans [Del19], l'auteur étudie un problème de stabilisation d'un modèle fluide-structure couplant les équations de Navier-Stokes décrivant la dynamique du fluide, et une équation différentielle ordinaire décrivant la dynamique de la structure. Dans ce cas, le contrôle qui agit sur la structure, est de dimension finie en boucle fermée.

• **Présentation de nos résultats**

La principale contribution de ce chapitre est la preuve de la stabilisation exponentielle du système (1.3.39), localement autour d'une solution stationnaire. Avant d'énoncer les résultats de ce chapitre, nous précisons que nous utiliserons la notation déjà introduite dans la Section 1.3.2, à la différence que, lorsqu'il est fait mention d'un certain T arbitraire, dans ce contexte nous considérons $T = \infty$.

La première difficulté, et qui est intrisèque à l'étude des systèmes fluide-structure, est le fait que le domaine du fluide varie au cours du temps. Pour faire face à cette difficulté, nous définissons un difféomorphisme X entre le domaine de référence Ω et le domaine physique $\Omega_{\eta(t)}$ à l'instant t. Cette transformation, introduite dans le Chapitre 5 (voir (5.2.8)), est utilisée pour définir le changement de variables suivant :

$$\begin{aligned}
&\widehat{\mathbf{u}}(t,\mathbf{z}) = e^{\omega t}\left(\mathbf{u}(t,X(t,\mathbf{z})) - \mathbf{u}_s(\mathbf{z})\right), \ \widehat{p}(t,\mathbf{z}) = e^{\omega t}\left(p(t,X(t,\mathbf{z})) - p_s(\mathbf{z})\right),\\
&\widehat{\eta}_1(t,z_1) = e^{\omega t}\eta(t,z_1), \ \ \widehat{\eta}_2(t,z_1) = e^{\omega t}\partial_t\eta(t,z_1), \ \ \widehat{\mathbf{f}}(t) = e^{\omega t}\mathbf{f}(t), \ \ \widehat{f} = e^{\omega t} f,\\
&\widehat{\eta}_1^{\pm}(t,z_1) = e^{\omega t}\eta^{\pm}(t,z_1), \ \ \widehat{\eta}_2^{\pm}(t,z_1) = e^{\omega t}\partial_t\eta^{\pm}(t,z_1),\\
&\widehat{\mathbf{g}}_p(t,z) = e^{\omega t}\mathbf{g}_p(t,z), \ \ \widehat{\mathbf{u}}^0 = \mathbf{u}^0 - \mathbf{u}_s,
\end{aligned} \tag{1.3.41}$$

pour tout $(t,\mathbf{z}) \in (0,\infty)\times\Omega$, avec $\omega > 0$. Ainsi, nous obtenons que $(\widehat{\mathbf{u}},\widehat{p},\eta_1,\eta_2)$ satisfait le système

$$\begin{cases}
\partial_t\widehat{\mathbf{u}} - \operatorname{div}\sigma(\widehat{\mathbf{u}},p) + (\mathbf{u}_s\cdot\nabla)\widehat{\mathbf{u}} + (\widehat{\mathbf{u}}\cdot\nabla)\mathbf{u}_s - A_1\widehat{\eta}_1 - A_2\widehat{\eta}_2 - \omega\widehat{\mathbf{u}} = e^{-\omega t}\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}_1,\widehat{\eta}_2) \ \text{dans } Q^\infty,\\
\operatorname{div}\widehat{\mathbf{u}} = e^{-\omega t}\operatorname{div}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}},\widehat{\eta}_1) + A_3\widehat{\eta}_1 \ \text{ dans } Q^\infty,\\
\widehat{\mathbf{u}} = \widehat{\mathbf{g}}_p \ \text{ sur } \Sigma_i^\infty, \ \ \widehat{\mathbf{u}} = 0 \ \text{ sur } \Sigma_r^\infty\cup\Sigma_w^\infty,\\
\widehat{\mathbf{u}} = \widehat{\eta}_2\vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^\infty, \ \ \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \ \text{ sur } \Sigma_n^\infty,\\
\widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 \ \text{ dans } \Omega,\\
\partial_t\widehat{\eta}_1 - \widehat{\eta}_2 - \omega\widehat{\eta}_1 = 0 \ \text{ dans } (0,\infty)\times(0,\ell_s),\\
\partial_t\widehat{\eta}_2 + \alpha\Delta_s^2\widehat{\eta}_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\widehat{\eta}_2 - A_4\widehat{\eta}_1 - \omega\widehat{\eta}_2 = \gamma_s^+\widehat{p} - \gamma_s^-\widehat{p} + e^{-\omega t}\widehat{F}_s(\widehat{\mathbf{u}},\widehat{\eta}_\mathbf{1})\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \widehat{f} \ \text{ dans } (0,\infty)\times(0,\ell_s),\\
\widehat{\eta}_1 = 0 \ \text{ et } \ \partial_{z_1}\widehat{\eta}_1 = 0 \ \text{ sur } (0,\infty)\times\{0\},\\
\partial_{z_1}^2\widehat{\eta}_1 = 0 \ \text{ et } \ \partial_{z_1}^3\widehat{\eta}_1 = 0 \ \text{ sur } (0,\infty)\times\{\ell_s\},\\
\widehat{\eta}_1(0) = 0 \ \text{ et } \ \widehat{\eta}_2(0) = \eta_2^0 \ \text{ dans } (0,\ell_s),
\end{cases} \tag{1.3.42}$$

où les opérateurs linéaires A_1, A_2, A_3 et A_4 sont définis dans (5.2.12), tandis que $\widetilde{\mathbf{F}}_f$, $\widetilde{\mathbf{G}}_{\mathrm{div}}$ et $\widetilde{F}_s$ sont définis dans l'appendice A.

Avant d'énoncer les principaux résultats de ce chapitre, nous introduirons la notation et les espaces utilisés tout au long du cette section.

Soit $\alpha \in (0,\alpha^*)$ et $\delta \in (\delta^*,1)$. Considérons d'abord la classe

$$\begin{aligned}
&\mathbf{u} \in L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega_{\eta(\cdot)})) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega_{\eta(\cdot)}),\\
&p \in L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega_{\eta(\cdot)})),\\
&\eta \in L^2(0,\infty;H^4_{\{0,\ell_s\}}(0,\ell_s)) \cap H^2(0,\infty;L^2(0,\ell_s)).
\end{aligned} \tag{1.3.43}$$

En partant de la Définition 1.3.2 avec $T=\infty$, nous introduisons l'espace

$$\begin{aligned}
\mathcal{Z}_\infty = \Big(&L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))\Big)\\
&\times L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s)),
\end{aligned} \tag{1.3.44}$$

que nous munissons de la norme

$$\begin{aligned}
\|(\mathbf{u},p,\eta)\|_{\mathcal{Z}_\infty} &= \|\mathbf{u}\|_{L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta})\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0})}\\
&+ \|p\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_{\delta})} + \|\eta\|_{H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s))},
\end{aligned} \tag{1.3.45}$$

où

$$\|\eta\|_{H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s))} = \|\eta\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\eta_t\|_{H^{2,1}((0,\infty)\times(0,\ell_s))}.$$

Pour un $R>0$ donné, nous introduisons également l'ensemble

$$\begin{aligned}
B_\infty(R,\mathbf{u}_0,\eta_2^0) :=\Big\{&(\widehat{\mathbf{u}},\widehat{p},\eta) \in \mathcal{Z}_\infty \mid \|(\widehat{\mathbf{u}},\widehat{p},\eta)\|_{\mathcal{Z}_\infty} \le R,\ \ \eta \in E(0,\infty)\\
&\text{et } \widehat{\mathbf{u}}(0)=\mathbf{u}_0,\ \eta(0)=0,\ \eta_t(0)=\eta_2^0\Big\}.
\end{aligned} \tag{1.3.46}$$

Nos résultats reposent sur trois hypothèses. Commençons par décrire la première hypothèse, qui concerne la régularité des solutions du système stationnaire (1.3.38).

<u>**Hypothèse 1**</u>. Soit $\alpha \in (0,\alpha^*)$ et $\delta \in (\delta^*,1)$. Supposons que $\mathbf{g}_s \in \mathbf{H}(\Gamma_i)$ et que le système (1.3.38) admette une solution $(\mathbf{u}_s,p_s) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega) \times H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)$. (A1)

Les deux autres hypothèses A2 et A3 sont précisées dans la Section 5.7. Dans l'hypothèse A2, on énonce une condition concernant les spectres des opérateurs adjoints d'Oseen et de la structure. Plus précisément, on suppose que les parties du spectre, contenues dans le demi-plan $\{\lambda \in \mathbb{C} \mid \Re\lambda \ge \omega\}$, des opérateurs adjoints d'Oseen et de la structure sont disjointes. D'autre part, l'hypothèse A3 énonce une propiété de prolongement unique.

Nous sommes maintenant en mesure d'énoncer les résultats principaux de ce chapitre.

Theorem 1.3.6. *Soit $\alpha \in (0,\alpha^*)$. Supposons que les hypothèses A1, A2 et A3 sont satisfaites. Pour tout $\omega>0$, il existe une famille $(w_i)_{i=1}^{N_c} \subset H^2_{\{0\}}(0,\ell_s)$ et un opérateur*

$$\mathcal{K} \in \mathcal{L}(\mathbf{L}^2(\Omega) \times H^2_{\{0\}}(0,\ell_s) \times L^2(0,\ell_s), \mathbb{R}^{N_c}),$$

tels qu'il existe $R>0$ et $r>0$, tels que pour tout $(\widehat{\mathbf{u}}^0,\eta_2^0) \in \mathbf{H}^1(\Omega) \times H^2_{\{0\}}(0,\ell_s)$ et $\widehat{\mathbf{g}}_p \in H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))$ satisfaisant

$$\begin{aligned}
&\widehat{\mathbf{u}}^0 = \widehat{\mathbf{g}}_p(0,\cdot) \ \ \textit{sur } \Gamma_i,\ \ \widehat{\mathbf{u}}^0 = 0 \ \ \textit{sur } \Gamma_r\cup\Gamma_w,\\
&\widehat{\mathbf{u}}^0 = \eta_2^0(0,\cdot)\vec{\mathbf{e}}_2 \ \ \textit{sur } \Gamma_s,\ \ \operatorname{div}\widehat{\mathbf{u}}^0 = 0 \ \ \textit{dans } \Omega,
\end{aligned} \tag{1.3.47}$$

et

$$\|\widehat{\mathbf{u}}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^2_{\{0\}}(0,\ell_s)} + \|\widehat{\mathbf{g}}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))} \leq r, \tag{1.3.48}$$

le système (1.3.42) *muni du contrôle feedback*

$$\widehat{f} = \sum_{i=1}^{N_c} \mathcal{K}_i(\widehat{\mathbf{u}}, \widehat{\eta}, \widehat{\eta}_t) w_i, \quad \textit{avec } \mathcal{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_{N_c}),$$

admet une solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}) \in B_\infty(R, \mathbf{u}_0, \eta_2^0)$ *satisfaisant*

$$\|(\widehat{\mathbf{u}}(t,\cdot), \widehat{\eta}(t,\cdot), \widehat{\eta}_t(t,\cdot))\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)\times H^3(0,\ell_s)\times H^1(0,\ell_s)} \leq CR \ \textit{ pour tout } t > 0,$$

où $C > 0$ *dépend de* α *et* r.

En conséquence du résultat précédent, on obtient le théorème suivant.

Theorem 1.3.7. *Soit* $\alpha \in (0, \alpha^*)$. *Supposons que les hypthèses A1, A2 et A3 sont satisfaites. Pour tout* $\omega > 0$, *il existe une famille* $(w_i)_{i=1}^{N_c} \subset H^2_{\{0\}}(0, \ell_s)$ *et un opérateur*

$$\mathcal{K} \in \mathcal{L}(\mathbf{L}^2(\Omega) \times H^2_{\{0\}}(0,\ell_s) \times L^2(0,\ell_s), \mathbb{R}^{N_c}),$$

tels qu'il existe $r > 0$, *tel que pour tout* $(\mathbf{u}^0, \eta_2^0) \in \mathbf{H}^1(\Omega) \times H^2_{\{0\}}(0,\ell_s)$ *et* $e^{\omega t}\mathbf{g}_p \in H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))$ *satisfaisant*

$$\begin{aligned} &\mathbf{u}^0 - \mathbf{u}_s = \mathbf{g}_p(0,\cdot) \ \textit{ sur } \Gamma_i, \quad \mathbf{u}^0 = 0 \ \textit{ sur } \Gamma_r \cup \Gamma_w, \\ &\mathbf{u}^0 - \mathbf{u}_s = \eta_2^0(0,\cdot)\vec{\mathbf{e}}_2 \ \textit{ sur } \Gamma_s, \quad \operatorname{div} \mathbf{u}^0 = 0 \ \textit{ dans } \Omega, \end{aligned} \tag{1.3.49}$$

et

$$\|\mathbf{u}^0 - \mathbf{u}_s\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^2(0,\ell_s)} + \|e^{\omega t}\mathbf{g}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))} \leq r, \tag{1.3.50}$$

le système (1.3.39) *muni du contrôle feedback*

$$f = \sum_{i=1}^{N_c} \mathcal{K}_i(\mathbf{u} \circ X^{-1} - \mathbf{u}_s, \eta, \eta_t) w_i, \quad \textit{avec } \mathcal{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_{N_c}),$$

admet une solution $(\mathbf{u}, p, \eta)$ *appartenant à la classe* (1.3.43) *satisfaisant*

$$\left\|\Big(\mathbf{u}(t, X^{-1}(t,\cdot)) - \mathbf{u}_s, \eta(t,\cdot), \eta_t(t,\cdot)\Big)\right\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)\times H^3(0,\ell_s)\times H^1(0,\ell_s)} \leq Ce^{-\omega t} \ \textit{ pour tout } t > 0,$$

où la transformation $X : \Omega \longrightarrow \Omega_{\eta(t)}$ *est celle définie dans* (1.3.38), *et* $C > 0$ *dépend de* α *et* r.

Avant de décrire les éléments essentiels de la preuve du Théorème 1.3.6, commençons par présenter l'idée générale de la démonstration.

1) *Description générale de la preuve.*

L'approche utilisée est classique. La stratégie consiste d'abord à stabiliser le système linéarisé. Ensuite, en utilisant le loi de contrôle en boucle fermée obtenue à cette étape, nous montrons, à l'aide d'un argument de point fixe, qu'elle permet également de stabiliser le système non linéaire, à condition que certaines hypothèses appropriées sur les données initiales et aux bords soient satisfaites. Dans le contexte des modèles d'interaction fluide-structure, cette approche a été utilisée par Raymond dans [Ray10]. Une stratégie similaire a été adoptée, par exemple, dans [FNR19] et [MR17].

La première chose à observer est que les difficultées rencontrées dans l'étude de l'existence de solutions fortes persistent dans l'analyse du problème de stabilisation. Ces difficultés sont liées à la présence des termes non linéaires $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ et $\widehat{F}_s$ dans le système (1.3.42), ainsi que le fait de considérer des conditions aux limites libres sur une extrémité de la structure. Cependant, de nouvelles difficultés apparaissent, que nous détaillons ci-dessous.

– Analyse des problèmes de valeurs propres. Puisque le système est linéarisé autour d'une solution stationnaire non triviale, l'analyse du problème aux valeurs propres direct nécessite de traiter une contrainte algébrique de la forme suivante :
$$\operatorname{div} \mathbf{v} = A_3 \eta_1 \ \text{ dans } \Omega, \tag{1.3.51}$$
où l'opérateur linéaire A_3 n'est pas nécessairement nul. Dans le problème de stabilisation étudié dans [Ray10] (voir aussi [MR17]), on voit que cette difficulté n'est pas présente, puisque l'opérateur $A_3 \equiv 0$, du fait de la linéarisation effectuée autour de la solution stationnaire nulle.
– Équivalence entre les formulations sous forme d'EDP et sous forme d'opérateur. Étroitement liée à la difficulté mentionnée ci-dessus se trouve la nécessité, dans l'analyse spectrale du système linéarisé, d'établir l'équivalence entre les formulations en équations aux dérivées partielles et les formulations en termes d'opérateurs, tant que pour les problèmes aux valeurs propres directs que pour les adjoints. Cette considération a une importance particulière dans le cadre de simulations numériques, car dans cette contexte nous travaillons directement avec la formulation EDP.

La preuve du Théorème 1.3.6 peut être divisée en deux arguments principaux :

2) *Étude du système linéarisé*.

Nous considérons le système fluide-structure linearisé
$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, q) + (\mathbf{u}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega \mathbf{v} = \mathbf{F}_f \ \text{ dans } Q^\infty, \\ \operatorname{div} \mathbf{v} = A_3\eta_1 + \operatorname{div} \mathbf{G}_{\text{div}} \ \text{ dans } Q^\infty, \\ \mathbf{v} = \mathbf{g}_p \ \text{ sur } \Sigma_i^\infty, \quad \mathbf{v} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^\infty, \quad \mathbf{v} = 0 \ \text{ sur } \Sigma_r^\infty \cup \Sigma_w^\infty, \quad \sigma(\mathbf{v}, q)\mathbf{n} = 0 \ \text{ sur } \Sigma_n^\infty, \\ \partial_t \eta_1 - \eta_2 - \omega \eta_1 = 0 \ \text{ in } (0,\infty) \times (0,\ell_s), \\ \partial_t \eta_2 + \alpha \Delta^2 \eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}} \eta_2 - A_4 \eta_1 - \omega \eta_2 = -\gamma_s^+ q + \gamma_s^- q + F_s + f \ \text{ dans } (0,\infty) \times (0,\ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ sur } (0,\infty) \times \{0\} \ \text{ et } \ \partial_{x_1}^2 \eta_1 = 0, \partial_{x_1}^3 \eta_1 = 0 \ \text{ sur } (0,\infty) \times \{\ell_s\}, \\ \eta_1(0) = 0 \ \text{ et } \ \eta_2(0) = \eta_2^0 \ \text{ dans } (0,\ell_s). \end{cases} \tag{1.3.52}$$

Nous prouvons le théorème suivant :

Theorem 1.3.8. *Soit $\alpha \in (0, \alpha^*)$ et $\delta \in (\delta^*, 1)$. Supposons que les hypothèses A1, A2 et A3 sont satisfaites. Supposons que $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H^1_{\{0\}}(0,\ell_s)$, $\mathbf{F}_f \in L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$, $\mathbf{g}_p \in H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))$, $F_s \in L^2(0,\infty;L^2(0,\ell_s))$, et que $\mathbf{G}_{\text{div}}$ satisfait (5.8.6), (5.8.7), (5.8.8) et $\mathbf{G}_{\text{div}}|_{t=0} = 0$. Nous supposons aussi les conditions de compatibilité suivantes :*
$$\begin{cases} \mathbf{v}_0 = 0 \ \text{ sur } \Gamma_i, \quad \mathbf{v}_0 = 0 \ \text{ sur } \Gamma_r \cup \Gamma_w, \\ \mathbf{v}_0 = \eta_2(0,\cdot)\vec{\mathbf{e}}_2 \ \text{ sur } \Gamma_s, \quad \operatorname{div} \mathbf{v}_0 = 0 \ \text{ dans } \Omega, \\ (P\mathbf{v}_0, 0, \eta_2^0) \in [\mathcal{D}(\mathcal{A}), \mathbf{Z}]_{1/2}, \quad P\mathbf{v}_0 \in [\mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathbf{V}^0_{n,\Gamma_d}(\Omega)]_{1/2}. \end{cases} \tag{1.3.53}$$
Alors, le système (1.3.52) avec
$$f = \sum_{i=1}^{N_c} [\mathcal{K}(\mathbf{v}, \eta_1, \eta_2)^\top] w_i,$$
où l'opérateur $\mathcal{K} \in \mathcal{L}(\mathbf{H}_0, \mathbb{R}^{N_c})$ est défini en (5.7.14), admet une unique solution $(\mathbf{v}, p, \eta_1, \eta_2)$ appartenant à $L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \times L^2(0,\infty;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)) \times H^{4,2}((0,\infty) \times$

$(0,\ell_s)) \times H^{2,1}((0,\infty)\times(0,\ell_s))$.

Idée de la preuve. La preuve de ce résultat repose sur les trois mêmes éléments déjà mentionnés dans la preuve du Théorème 1.3.5 de la Section 1.3.2. Par conséquent, nous nous concentrerons sur les ingrédients de la preuve qui sont directement liés à la construction de la loi de contrôle.

(a) *Réduction de la stabilisabilité du système (1.3.52) à la stabilisabilité du système projeté.*

Le point de départ de l'analyse est le fait que la résolvante de l'opérateur fluide-structure $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ est compacte, et par conséquent le spectre de $\mathcal{A}$ est ponctuel. Nous avons alors la décomposition suivante :

$$\mathbf{Z} = \mathbf{Z}_u \oplus \mathbf{Z}_s \ \text{ et } \ \mathbf{Z}^* = \mathbf{Z}_u^* \oplus \mathbf{Z}_s^*,$$

où

$$\mathbf{Z}_u = \bigoplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \ \text{ et } \ \mathbf{Z}_u^* = \bigoplus_{j\in J_u} G_{\mathbb{R}}^*(\lambda_j),$$

avec $J_u := \{j \in \mathbb{N}^* \mid \Re\lambda_j \geq -\omega\}$ et

$$G_{\mathbb{R}}(\lambda_j) = \operatorname{span}\{\Re G_{\mathbb{C}}(\lambda_j) \cup \Im G_{\mathbb{C}}(\lambda_j)\} \text{ et } G_{\mathbb{R}}^*(\lambda_j) = \operatorname{span}\{\Re G_{\mathbb{C}}^*(\lambda_j) \cup \Im G_{\mathbb{C}}^*(\lambda_j)\}$$

désignant les espaces propres généralisés réels. Ici, les sous-espaces $\mathbf{Z}_u$ et $\mathbf{Z}_u^*$ sont invariants sous $(e^{t\mathcal{A}})_{t\geq 0}$ et $(e^{t\mathcal{A}^*})_{t\geq 0}$, respectivement. Ensuite, nous désignons par Π_u la projection de $\mathbf{Z}$ sur $\mathbf{Z}_u$ selon $\mathbf{Z}_s$ et par Π_s la projection de $\mathbf{Z}$ sur $\mathbf{Z}_s$ selon $\mathbf{Z}_u$.

La Proposition 5.7.2 nous permet de caractériser l'opérateur de projection Π_u. Plus précisément, ce résultat montre l'existence de bases bi-orthogonales $\{(P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})\}_{1\leq i\leq N_u}$ et $\left\{M^*(P\mathbf{\Phi}_i, \zeta_{1,i}, \zeta_{2,i})^\top\right\}_{1\leq i\leq N_u}$ de $\mathbf{Z}_u$ et $\mathbf{Z}_u^*$ respectivement, telles que

$$\Pi_u(\mathbf{v},\eta_1,\eta_2)^\top = \sum_{i=1}^{N_u} \left\langle (\mathbf{v},\eta_1,\eta_2)^\top, M^*(P\mathbf{\Phi}_i,\zeta_{1,i},\zeta_{2,i})^\top \right\rangle_{\mathbf{Z},\mathbf{Z}'} (P\mathbf{v}_i,\eta_{1,i},\eta_{2,i})^\top,$$

pour tout $(\mathbf{v},\eta_1,\eta_2)^\top \in \mathbf{Z}$. Ici, P désigne l'opérateur de Leray, $N_u = \dim(\mathbf{Z}_u)$ et M est la matrice de masse définie en (5.3.37).

Considérons le système

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div}\sigma(\mathbf{v},q) + (\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega\mathbf{v} = 0 \ \text{ dans } Q^\infty, \\ \operatorname{div}\mathbf{v} = A_3\eta_1 \ \text{ dans } Q^\infty, \\ \mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^\infty, \ \ \mathbf{v} = 0 \ \text{ sur } \Sigma_d^\infty \setminus \Sigma_s^\infty, \ \ \sigma(\mathbf{v},q)\mathbf{n} = 0 \ \text{ sur } \Sigma_n^\infty, \\ \partial_t\eta_1 - \eta_2 - \omega\eta_1 = 0 \ \text{ dans } (0,\infty)\times(0,\ell_s), \\ \partial_t\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 - A_4\eta_1 - \omega\eta_2 = -\gamma_s^+ q + \gamma_s^- q \\ \qquad\qquad + \sum_{i=1}^{N_c} f_i w_i \ \text{ dans } (0,\infty)\times(0,\ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ sur } (0,\infty)\times\{0\} \ \text{ et } \ \partial_{x_1}^2\eta_1 = 0, \partial_{x_1}^3\eta_1 = 0 \ \text{ sur } (0,\infty)\times\{\ell_s\}. \end{cases} \tag{1.3.54}$$

Nous avons que le système (1.3.54) peut être réécrit de la manière suivante :

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A}+\omega I)\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\mathbf{f}, \ \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} P\mathbf{v}^0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \\ (I-P)\mathbf{v} = \nabla N_{\mathrm{div}} A_3\eta_1 - \nabla N_s\eta_2, \end{cases} \tag{1.3.55}$$

où $\mathcal{B}\mathbf{f} = \sum_{i=1}^{N_c} f_i(0, 0, (I + \gamma_s^{+,-} N_s)^{-1} w_i)^\top$.

Proposition 1.3.3. *Le triplet $(P\mathbf{v}, \eta_1, \eta_2)^\top \in \mathcal{D}(\mathcal{A})$ est la solution de la première équation de (1.3.55), si et seulement si,*

$$\zeta_u = \left(\left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1 \leq j \leq N_u} \qquad et \quad (\mathbf{v}_s, \eta_{1,s}, \eta_{2,s}) := \Pi_s (P\mathbf{v}, \eta_1, \eta_2)^\top,$$

satisfont

$$\begin{cases} \dfrac{d}{dt}\zeta_u = (\Lambda_u + \omega I_{\mathbb{R}^{N_u}})\zeta_u + \mathcal{B}_u \mathbf{f}, \quad \zeta_u(0) = \zeta_u^0, \\ \dfrac{d}{dt}\begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix} = \mathcal{A}_{s,\omega} \begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix} + \mathcal{B}_s \mathbf{f}, \quad \begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix}(0) = \Pi_s \begin{pmatrix} \mathbf{v}^0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \end{cases} \tag{1.3.56}$$

où la matrice Λ_u est donnée par

$$\Lambda_u = [\Lambda_{i,j}]_{1 \leq i,j \leq N_u}, \quad \Lambda_{i,j} = \left\langle \mathcal{A}(P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top, M^*(P\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})^\top \right\rangle_{\mathbf{Z},\mathbf{Z}'}$$

et

$$\zeta_u^0 = \left(\left\langle \begin{pmatrix} P\mathbf{v}^0 \\ 0 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1 \leq j \leq N_u} \qquad et \quad \mathcal{B}_s = \Pi_s \mathcal{B}.$$

Étant donné que l'opérateur $\mathcal{A}_{s,\omega} = \Pi_s(\mathcal{A} + \omega I)$ satisfait

$$\|e^{t\mathcal{A}_{s,\omega}}\|_{\mathcal{L}(\mathbf{Z})} \leq Ce^{-\varepsilon_s t} \quad \forall t > 0, \quad 0 < \varepsilon_s < \mathrm{dist}(\Re\,\mathrm{spec}(\mathcal{A}_{s,\omega}), 0),$$

on voit que la stabilisabilité du système linéarisé se réduit à la stabilisabilité du couple $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$.

(b) <u>*Stabilisabilité du couple $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$*</u>.

Theorem 1.3.9. *Supposons que les hypothèses A1, A2 et A3 sont satisfaites et que les fonctions $(w_i)_{1 \leq i \leq N_c}$ sont donné par (5.7.3). Alors, le couple $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ est stabilisable.*

<u>*Idée de la preuve*</u>. Le cœur de la preuve est l'utilisation de [BDDM07, Proposition 3.3, p.492] qui nous permet d'établir que le couple $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ est stabilisable, si et seulement si,

$$\ker(\lambda I - \mathcal{A}^*) \cap \ker(\mathcal{B}^*) = \{0\} \text{ pour tout } \lambda \in \mathbb{C} \text{ tel que } \Re\lambda \geq -\omega.$$

C'est précisément à ce stade que nous utilisons les hypothèses A2 et A3.

(c) <u>*Loi de contrôle feedback*</u>.

Grâce au Théorème 1.3.9 et au fait que $-(\Lambda_u + \omega I_{\mathbb{R}^{N_u}})$ est stable, il découle de [KR09, Theorem 3] que l'équation de Riccati (de petite dimension)

$$\begin{aligned} & Q_u \in \mathcal{L}(\mathbb{R}^{N_u}), \quad Q_u = Q_u^\top > 0, \\ & (\Lambda_u^\top + \omega I_{\mathbb{R}^{N_u}})Q_u + Q_u(\Lambda_u + \omega I_{\mathbb{R}_{N_u}}) - Q_u \mathcal{B}_u \mathcal{B}_u^\top Q_u = 0 \end{aligned} \tag{1.3.57}$$

admet une unique solution. Nous introduisons maintenant l'opérateur $\mathcal{K}_p \in \mathcal{L}(\mathbf{Z}_0, \mathbb{R}^{N_c})$, défini par

$$\mathcal{K}_p \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \left(\sum_{j=1}^{N_u} K_u^{i,j} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1 \le i \le N_c} . \tag{1.3.58}$$

On a donc :

Theorem 1.3.10. *Sous les hypothèses A1, A2 et A3, l'opérateur $\mathcal{K}_p$ introduit en (1.3.58) fournit une loi de contrôle pour le couple $(\mathcal{A}+\omega I, \mathcal{B})$. De plus, l'opérateur $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p$, avec domaine le $D(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p) = D(\mathcal{A})$, est le générateur infinitésimal d'un semi-groupe analytic exponentiellement stable sur $\mathbf{Z}$.*

(d) *Utilisation de la loi de contrôle $\mathcal{K}$ dans le système linéarisé non homogène (1.3.52).*

La relation

$$\mathcal{K}_p \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{K} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} \tag{1.3.59}$$

(voir Proposition 5.7.4) nous permet d'établir le lien entre la loi de contrôle $\mathcal{K}_p$ et la loi de contrôle $\mathcal{K}$. Cette relation peut être comprise comme le lien entre "la loi de contrôle dans l'équation d'opérateurs et la loi de contrôle dans l'équation aux dérivées partielles (EDP)", nous permet de travailler directement avec l'EDP.

3) *Estimations des termes non linéaires et argument de point fixe.*

La dernière étape de la preuve consiste à estimer les termes non linéaires et à utiliser un argument de point fixe. À cette fin, nous définissons l'application $\mathcal{N} : B_\infty(R, \mathbf{u}_0, \eta_2^0) \longrightarrow B_\infty(R, \mathbf{u}_0, \eta_2^0)$ par

$$\mathcal{N}(\widehat{\mathbf{\Phi}}, \widehat{\psi}, \widehat{k}) = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}) \quad \text{pour tout } (\widehat{\mathbf{\Phi}}, \widehat{\psi}, \widehat{k}) \in B_\infty(R, \mathbf{u}_0, \eta_2^0),$$

où $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta})$ est solution du système

$$\begin{cases} \partial_t \widehat{\mathbf{u}} - \operatorname{div} \sigma(\widehat{\mathbf{u}}, \widehat{p}) - A_1 \widehat{\eta} - A_2 \widehat{\eta}_t = e^{-\omega t} \widehat{\mathbf{F}}_f(\widehat{\mathbf{\Phi}}, \widehat{\psi}, \widehat{k}) \ \text{ dans } Q^\infty, \\ \operatorname{div} \widehat{\mathbf{u}} = A_3 \widehat{\eta}_1 + e^{-\omega t} \operatorname{div} \widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{\Phi}}, \widehat{k}) \ \text{ dans } Q^\infty, \\ \widehat{\mathbf{u}} = \widehat{\mathbf{g}}_p \ \text{ sur } \Sigma_i^\infty, \ \ \widehat{\mathbf{u}} = 0 \ \text{ sur } \Sigma_w^\infty \cup \Sigma_r^\infty, \ \ \widehat{\mathbf{u}} = \widehat{\eta}_t \vec{\mathbf{e}}_2 \ \text{ sur } \Sigma_s^\infty, \\ \sigma(\widehat{\mathbf{u}}, \widehat{p}) \mathbf{n} = 0 \ \text{ sur } \Sigma_n^\infty, \ \ \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 \ \text{ dans } \Omega, \\ \partial_t^2 \widehat{\eta} + \alpha \Delta_s^2 \widehat{\eta} + \gamma (\Delta_s^2)^{\frac{1}{2}} \widehat{\eta}_t = -\gamma_s^{+,-} \widehat{p} + e^{-\omega t} \widehat{F}_s(\widehat{\mathbf{\Phi}}, \widehat{k}) + \sum_{i=1}^{N_c} \mathcal{K}_i(\widehat{\mathbf{u}}, \widehat{\eta}, \widehat{\eta}_t) w_i \ \text{ dans } (0,\infty) \times (0, \ell_s), \\ \widehat{\eta} = 0 \ \text{ et } \ \partial_{x_1} \widehat{\eta} = 0 \ \text{ sur } (0,\infty) \times \{0\}, \\ \partial_{x_1}^2 \widehat{\eta} = 0 \ \text{ et } \ \partial_{x_1}^3 \widehat{\eta} = 0 \ \text{ sur } (0,\infty) \times \{\ell_s\}, \\ \widehat{\eta}(0) = 0 \ \text{ et } \ \widehat{\eta}_t(0) = \eta_2^0 \ \text{ dans } (0, \ell_s). \end{cases} \tag{1.3.60}$$

En prenant $r > 0$ et $R > 0$ de manière appropriée, on peut démontrer que l'application $\mathcal{N}$ est bien définie, et qu'elle est une contraction stricte. Finalement, l'existence d'un unique point fixe de $\mathcal{N}$ découle du Théorème du point fixe de Banach. Ici, les estimations présentées dans les Lemmes 5.9.2, 5.9.3 et 5.9.4 sont essentielles.

1.3.5 Chapitre 6 : Simulations numériques du problème de stabilisation du système d'interaction fluide-structure

Tout au long de cette section, nous considérons la configuration geométrique représentée dans la Figure 1.2, dont la description précise a été introduite dans la Section 1.2 (voir (1.2.1), (1.2.2) et (1.2.4)).

• Formulation du problème et littérature existante

Dans ce chapitre on s'interesse à la simulation numérique de la stabilisation du système d'interaction fluide-structure

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \;\text{ dans } Q_\eta^\infty, \\ \operatorname{div}\mathbf{u} = 0 \;\text{ dans } Q_\eta^\infty, \\ \mathbf{u} = \mathbf{g}_s^i + \mathbf{g}_p^i \;\text{ sur } \Sigma_i^\infty, \;\; \mathbf{u} = 0 \;\text{ sur } \Sigma_w^\infty, \;\; \mathbf{u} = 0 \;\text{ sur } \Sigma_r^\infty, \\ \mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \;\text{ sur } \Sigma_\eta^\infty, \;\; \sigma(\mathbf{u},p)\mathbf{n} = 0 \;\text{ sur } \Sigma_n^\infty, \\ \mathbf{u}(0) = \mathbf{u}^0 \;\text{ dans } \Omega, \\ \partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) + f_s + f \text{ dans } (0,\infty)\times(0,\ell_s), \\ \eta = 0 \;\text{ et }\; \partial_{x_1}\eta = 0 \;\text{ sur } (0,\infty)\times\{0\}, \\ \partial_{x_1}^2\eta = 0 \;\text{ et }\; \partial_{x_1}^3\eta = 0 \;\text{ sur } (0,\infty)\times\{\ell_s\}, \\ \eta(0) = 0 \;\text{ et }\; \partial_t\eta(0) = \eta_2^0 \;\text{ dans } (0,\ell_s), \end{cases} \tag{1.3.61}$$

à l'aide d'un contrôle f qui agit sur l'équation de la structure. Avant de décrire le problème plus en détail, introduisons quelques notations.

Dans le système (1.3.61),

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}_{\eta(t)}^+ + \sigma^-(\mathbf{u},p)\mathbf{n}_{\eta(t)}^-\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2,$$

avec

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x_1,\eta(t,x_1)\pm e), p(t,x_1,\eta(t,x_1)\pm e)).$$

L'opérateur d'amortissement B est donné par

$$B = \Delta_s^2 = \partial_{x_1}^4, \quad D(B) = H^4_{\{0,\ell_s\}}(0,\ell_s).$$

La condition au bord en entrée $\mathbf{g}_s^i$ est indépendante du temps, tandis que $\mathbf{g}_p^i$ est une perturbation de $\mathbf{g}_s^i$ (dépendante du temps).

Considérons le système stationnaire

$$\begin{cases} (\mathbf{u}_s\cdot\nabla)\mathbf{u}_s - \operatorname{div}\sigma(\mathbf{u}_s,p_s) = 0, \;\; \text{dans } \Omega, \\ \operatorname{div}\mathbf{u}_s = 0 \;\text{ dans } \Omega, \\ \mathbf{u}_s = \mathbf{g}_s^i, \;\; \text{on } \Gamma_i, \;\; \mathbf{u}_s = 0 \;\text{ sur } \Gamma\setminus\Gamma_i, \\ \sigma(\mathbf{u}_s,p_s)\mathbf{n} = 0 \;\text{ sur } \Gamma_n. \end{cases} \tag{1.3.62}$$

La fonction f_s est indépendante du temps et est choisie de telle sorte que le triplet $(\mathbf{u},\eta,\eta_t) = (\mathbf{u}_s,0,0)$ constitue une solution stationnaire du système (1.3.39). Ainsi,

$$f_s = \sigma(\mathbf{u}_s,p_s)\mathbf{n}^+\cdot\vec{\mathbf{e}}_2 + \sigma(\mathbf{u}_s,p_s)\mathbf{n}^-\cdot\vec{\mathbf{e}}_2, \tag{1.3.63}$$

où $\mathbf{n}^+$ (respectivement $\mathbf{n}^-$) représente la normale unitaire exterièure à Γ_s^+ (respectivement Γ_s^-). Nous supposons aussi que la fonction f (qui joue le rôle de contrôle) s'écrit sous la forme

$$f = \sum_{j=1}^{N_c} f_j(t) w_j(z_1), \tag{1.3.64}$$

où les fonctions w_j sont choisies de manière appropriée.

L'objectif de ce chapitre est de simuler numériquement le problème de stabilisation suivant :

Trouver un contrôle f donné sous la forme de boucle fermée, capable de stabiliser le système (1.3.61) autour de la solution stationnaire $(\mathbf{u}, \eta, \eta_t) = (\mathbf{u}_s, 0, 0)$, à condition que $\mathbf{g}_p^i$, $\mathbf{u}^0 - \mathbf{u}_s$ et η_2^0 soient suffisamment petits dans certains espaces fonctionnels appropriés.

En plus de la difficulté liée au fait que le domaine du fluide change au cours du temps, il y a maintenant la difficulté supplémentaire du cacul du contrôle. Un premier travail qui aborde ces problèmes conjointement, traitant un problème de stabilisation d'un système d'interaction fluide-structure, se trouve dans [FNR] (voir aussi [Ndi16]). Dans ces travaux, le calcul numérique du contrôle feedback est basé sur l'utilisation d'une équation de Riccati de petite dimension, grâce à une projection du système sur l'espace inestable. Une particularité de ces travaux est que les simulations numériques sont menées dans un domaine fixe. Le modèle d'interaction fluide-structure considéré dans ces travaux, est un système couplant les équations incompressible de Navier-Stokes et une structure modélisée par une équation d'Euler-Bernoulli avec des conditions encastrées. D'autre part, dans [Del18], une étude d'un problème de stabilisation d'un système couplant les équations incompressible de Navier-Stokes et une équation différentielle ordinaire modélisant la structure est effectuée. Dans ce travail, une stratégie similaire à celle utilisée dans [FNR] est mise en œuvre pour construire le contrôle feedback. La principale différence entre les deux travaux est que, contrairement à l'approche utilisée dans [FNR], où tous les calculs du problème en évolution sont effectués dans un domaine de référence fixe, dans [Del18] la résolution du problème d'évolution est menée dans le domaine physique en utilisant la méthode de domaines fictifs.

• **<u>Présentation de nos résultats</u>**

La stratégie utilisée dans ce chapitre diffère de celle utilisée dans [FNR]. Pour comprendre les différences entre eux, commençons par mentionner les principaux éléments de [FNR] :

(1) Le premier aspect important est que, pour réécrire le système d'interaction fluide-structure dans le domaine de référence, les auteurs utilisent une tranformation géométrique explicite. Ensuite, la linéarisation du système résultant est effectuée "manuellement". Puis, le calcul de la loi de contrôle est realisé dans le domaine de référence Ω.

(2) Le deuxième point important est qu'une fois la loi de contrôle pour le système linéarisé est calculée dans le domaine de référence fixe Ω, elle est injectée dans le système non linéaire, qui est alors résolu dans le domaine de référence.

Comparons maintenant la stratégie adoptée dans [FNR] avec celle utilisée dans ce chapitre.

(1*b*) Pour réécrire le système d'interaction fluide-structure dans le domaine de référence, nous n'utilisons pas la transformation géométrique comme dans [FNR]. Au lieu de cela, nous considérons une transformation définie en termes d'une extension harmonique dans Ω de la trace du déplacement de la structure. Ensuite, une fois le système non linéaire reformulé dans sa forme faible sur le domaine de référence fixe Ω, la linéarisation est effectuée à l'aide d'une routine fournie par la bibliothèque GetFEM++. De manière similaire à l'approche utilisée dans [FNR], la loi de contrôle est calculée sur le domaine de référence fixe Ω.

(2*b*) De manière analogue à l'approche adoptée dans [FNR], la loi de contrôle calculée sur le domaine de référence fixe Ω est appliquée au système non linéaire, qui contrairement à [FNR], est résolu sur le domaine physique $\Omega_{\eta(t)}$ en utilisant l'algorithme 2.

En résumé, les principales différences entre les stratégies utilisées dans [FNR] et celles utilisé dans ce chapitre sont les transformations permettant de réécrire le système dans le domaine de référence et l'approche pour résoudre le problème direct différent dans le deux cas.

Nous commençons par rappeler la stratégie utiilsée pour faire face au fait que le domaine peut changer au cours du temps.

Nous considérons d'abord la transformation ALE $\mathcal{A}(t,\cdot):\Omega_{ref}\longrightarrow\Omega_{\eta(t)}$ définie par

$$\mathcal{A}(t,\cdot)=I+\int_0^t \mathbf{w}(s,\cdot)\,ds, \tag{1.3.65}$$

où $\mathbf{w}(t,\cdot)$ est solution de l'équation elliptique

$$\Delta\mathbf{w}=0 \ \text{ dans } \Omega,\ \ \mathbf{w}=\mathbf{u}|_{\Gamma_s} \ \text{ sur } \Gamma_s,\ \ \mathbf{w}=0 \ \text{ sur } \Gamma\setminus\Gamma_s. \tag{1.3.66}$$

Ici, $\mathbf{u}|_{\Gamma_s}$ représente la trace de la vitesse du fluide sur Γ_s.

La discrétisation en temps est traitée à l'aide de la méthode classique d'Euler implicite. Nous désignons par Δt le pas de temps et $t^k=k\Delta t$, pour $k\in\mathbb{N}$ le niveau de temps k. Pour tout $k\in\mathbb{N}$, $\Omega^k:=\mathcal{A}(t^k,\Omega_{ref})$ avec frontière $\Gamma^k=\Gamma_i\cup\Gamma_0\cup\Gamma_s^k\cup\Gamma_n$, où $\Gamma_0=\Gamma_r\cup\Gamma_w$ et $\Gamma_s^k=\Gamma^k_{s,top}\cup\Gamma^k_{s,bot}\cup\Gamma^k_{s,lat}$ (voir Figure 1.4). Nous désignons par $\mathbf{u}^k$, p^k et $\boldsymbol{\lambda}^k$ les approximations de $\mathbf{u}(t^k,\cdot)$, $p(t^k,\cdot)$ et $\boldsymbol{\lambda}(t^k,\cdot)$, respectivement. Ici, $\boldsymbol{\lambda}^k=(\boldsymbol{\lambda}_i^k,\boldsymbol{\lambda}_0^k,\boldsymbol{\lambda}^k_{s,top},\boldsymbol{\lambda}^k_{s,bot},\boldsymbol{\lambda}^k_{s,lat})^\top$ représentent les multiplicateurs de Langrange associés aux conditions au bord de Dirichlet. Nous notons également η_1^k et η_2^k l'approximation de $\eta_1(t^k,\cdot)$ et $\eta_2(t^k,\cdot)$ définie sur $(0,\ell_s)$, respectivement.

Avant d'introduire l'algorithme de résolution, nous allons considérer le problème intermédiaire suivant. Étant donné $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k et η_2^k, considérons le problème suivant :

Trouver $\widehat{\mathbf{u}}^{k+1}\in\mathbf{H}^1(\Omega^k)$, $\widehat{p}^{k+1}\in L^2(\Omega^k)$, $\widehat{\boldsymbol{\lambda}}^{k+1}\in\mathbf{H}^{-\frac12}(\Gamma^k\setminus\Gamma_n)$, $\eta_1^{k+1},\eta_2^{k+1}\in H^2_{\{0\}}(0,\ell_s)$ tels que

$$\left\{\begin{aligned}
&\int_{\Omega^k}\frac{\widehat{\mathbf{u}}^{k+1}-\mathbf{u}^k}{\Delta t}\cdot\boldsymbol{\phi}=a_f(\widehat{\mathbf{u}}^{k+1},\boldsymbol{\phi})+b(\boldsymbol{\phi},\widehat{p}^{k+1})+c(\widehat{\mathbf{u}}^{k+1}-\mathbf{w}^k,\widehat{\mathbf{u}}^{k+1},\boldsymbol{\phi})\\
&\qquad\qquad+\int_{\Gamma_i}\widehat{\boldsymbol{\lambda}}_i^{k+1}\cdot\boldsymbol{\phi}+\int_{\Gamma_0^k}\widehat{\boldsymbol{\lambda}}_0^{k+1}\cdot\boldsymbol{\phi}+\int_{\Gamma^k_{s,top}}\widehat{\boldsymbol{\lambda}}_{s,top}^{k+1}\cdot\boldsymbol{\phi}\\
&\qquad\qquad+\int_{\Gamma^k_{s,bot}}\widehat{\boldsymbol{\lambda}}_{s,bot}^{k+1}\cdot\boldsymbol{\phi},+\int_{\Gamma^k_{s,lat}}\widehat{\boldsymbol{\lambda}}_{s,lat}^{k+1}\cdot\boldsymbol{\phi},\ \ \forall\boldsymbol{\phi}\in\mathbf{H}^1(\Omega^k),\\
&b(\widehat{\mathbf{u}}^{k+1},\psi)=0,\ \ \forall\psi\in L^2(\Omega^k),\\
&\int_{\Gamma_i}\widehat{\mathbf{u}}^{k+1}\cdot\boldsymbol{\tau}=\int_{\Gamma_i}\mathbf{g}^i\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma_i),\qquad\int_{\Gamma_0}\widehat{\mathbf{u}}^{k+1}\cdot\boldsymbol{\tau}=0,\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma_0),\\
&\int_{\Gamma^k_{s,top}}\widehat{\mathbf{u}}^{k+1}\cdot\boldsymbol{\tau}=\int_{\Gamma^k_{s,top}}\eta_2^{k+1}\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^k_{s,top}),\\
&\int_{\Gamma^k_{s,bot}}\widehat{\mathbf{u}}^{k+1}\cdot\boldsymbol{\tau}=\int_{\Gamma^k_{s,bot}}\eta_2^{k+1}\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^k_{s,bot}),\\
&\int_{\Gamma^k_{s,lat}}\widehat{\mathbf{u}}^{k+1}\cdot\boldsymbol{\tau}=\int_{\Gamma^k_{s,lat}}\eta_2^{k+1}\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^k_{s,lat}),\\
&\int_0^{\ell_s}\frac{\eta_1^{k+1}-\eta_1^k}{\Delta t}\zeta=\int_0^{\ell_s}\eta_2^{k+1}\zeta,\ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
&\int_0^{\ell_s}\frac{\eta_2^{k+1}-\eta_2^k}{\Delta t}\zeta=a_s^1(\eta_1^{k+1},\zeta)+a_s^2(\eta_2^{k+1},\zeta)\\
&\qquad\qquad-\int_0^{\ell_s}\widehat{\boldsymbol{\lambda}}_{s,top}^{k+1}\cdot\vec{\mathbf{e}}_2\zeta\sqrt{1+(\eta^k_{1,x})^2}\\
&\qquad\qquad-\int_0^{\ell_s}\widehat{\boldsymbol{\lambda}}_{bot}^{k+1}\cdot\vec{\mathbf{e}}_2\zeta\sqrt{1+(\eta^k_{1,x})^2}\\
&\qquad\qquad+\sum_{j=1}^{N_c}\int_0^{\ell_s}w_j(z_1)\zeta(z_1)f_j^k(t)+\int_0^{\ell_s}f_s\zeta,\ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s).
\end{aligned}\right. \tag{1.3.67}$$

où

$$a_f(\mathbf{v},\boldsymbol{\phi}) = -2\nu\int_{\Omega^k}\varepsilon(\mathbf{v}):\varepsilon(\boldsymbol{\phi}),\quad b(\boldsymbol{\phi},q)=\int_{\Omega^k}(\operatorname{div}\boldsymbol{\phi})\,q,\quad c(\mathbf{v},\mathbf{v},\boldsymbol{\phi})=\int_{\Omega^k}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\phi},$$
$$a_s^1(\eta_1,\zeta) = -\alpha\int_0^{\ell_s}\Delta\eta_1\cdot\Delta\zeta,\qquad a_s^2(\eta_2,\zeta) = -\gamma\int_0^{\ell_s}\Delta\eta_2\cdot\Delta\zeta.$$

Algorithm 2: Semi-implicit algorithm

Pour $k\geq 1$:

1 : Résoudre le système linéaire obtenu après avoir appliqué l'algorithme de Newton au système (1.3.35) pour obtenir $\widehat{\mathbf{u}}^{k+1}$, $\widehat{p}^{k+1}$, $\widehat{\boldsymbol{\lambda}}^{k+1}$, η_1^{k+1}, η_2^{k+1}.

2 : Calculer la vitesse du maillage $\widehat{\mathbf{w}}^{k+1}:\Omega^k\to\mathbb{R}^2$ satisfaisant l'équation elliptique

$$\begin{cases}\Delta\widehat{\mathbf{w}}^{k+1}=0 \ \text{ dans } \Omega^k,\\ \widehat{\mathbf{w}}^{k+1}=\widehat{\mathbf{u}}^{k+1} \ \text{ sur } \Gamma_s^k,\\ \widehat{\mathbf{w}}^{k+1}=0 \ \text{ sur } \Gamma\setminus\Gamma_s^k.\end{cases}\tag{1.3.68}$$

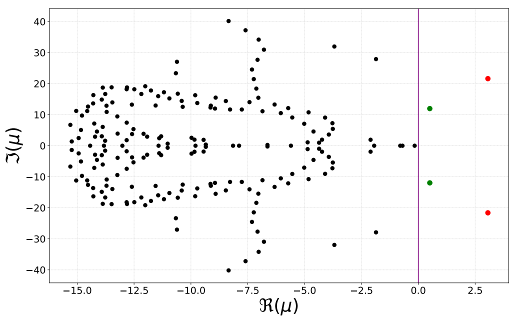
3 : Définir $\mathcal{A}^k(\widehat{\mathbf{x}}) := \widehat{\mathbf{x}}+\Delta t\widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}})$ et $\Omega^{k+1}:=\mathcal{A}^k(\Omega^k)$.

4 : Définir $\mathbf{u}^{k+1}:\Omega^{k+1}\to\mathbb{R}^2$, $p:\Omega^{k+1}\to\mathbb{R}$, $\boldsymbol{\lambda}^{k+1}:\Omega^{k+1}\to\mathbb{R}^2$ et $\mathbf{w}^{k+1}:\Omega^{k+1}\to\mathbb{R}^2$ par

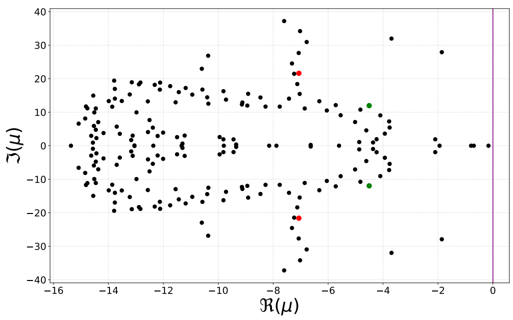
$$\mathbf{u}^{k+1}(\mathbf{x})=\widehat{\mathbf{u}}^{k+1}(\widehat{\mathbf{x}}),\ \ p^{k+1}(x)=\widehat{p}^{k+1}(\widehat{\mathbf{x}}),\ \ \boldsymbol{\lambda}^{k+1}(x)=\widehat{\boldsymbol{\lambda}}^{k+1}(\widehat{\mathbf{x}})$$
$$\text{and }\ \mathbf{w}^{k+1}(\mathbf{x})=\widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}}),\ \ \forall\mathbf{x}=\mathcal{A}^k(\widehat{\mathbf{x}}),\,\widehat{\mathbf{x}}\in\Omega^k.$$

1. **Comparaison entre les spectres de l'opérateur fluide-structure**.

Un premier élément à analyser est de comparer les spectres des systèmes sans et avec la loi de contrôle. Dans ce cadre, nous analysons les spectres d'un système en fixant le nombre de Reynolds $Re = 200$ ainsi que le coefficient d'amortissement $\gamma = 10^{-6}$ de la structure. Par exemple, dans la Figure 1.10 nous montrons une partie du spectre du sytème avec et sans loi de contrôle dans cette configuration.



(a) Spectre de l'opérateur A.

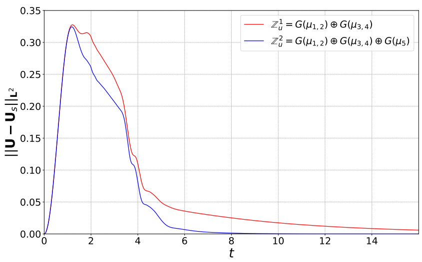


(b) Spectre de l'opérateur $A - BK$.

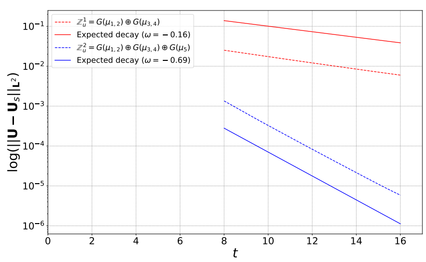
FIGURE 1.10 – Comparaison du spectre fluide-structure correspondante à un nombre de Reynolds$Re = 200$, coefficient de rigidité $\boldsymbol{\alpha = 10^{-1}}$, et un coefficient d'amortissement $\gamma = 10^{-6}$, avec et sans loi de contrôle.

2. **Influence du sous-espace instable $\mathbb{Z}_u$.**

Un deuxième élément intéressant à étudier est la façon dont le choix du sous-espace $\mathbb{Z}_u$ affecte la performance du contrôle. Dans le même esprit que pour l'analyse précédente, nous fixons le nombre de Reynolds à $Re = 200$, le coefficient d'amortissement $\gamma = 10^{-6}$ et l'amplitude de la perturbation $\beta = 1.5$, et ensuite, nous faisons varier le coefficient de rigidité α. Par exemple, dans le cas où le coefficient de rigidité $\alpha = 10^{-1}$, la Figure 1.11 présente l'évolution des normes $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$ et des taux de décroissance en fonction de deux contrôles : l'un basé sur le sous-espace instable $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ et l'autre sur $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$.

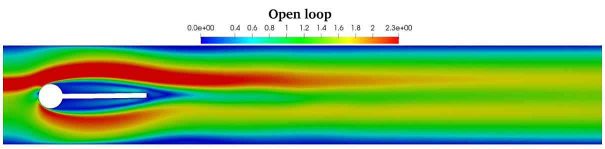


(a) Évolution de $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$

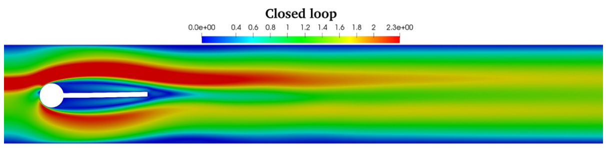


(b) Évolution de $\log(\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2})$

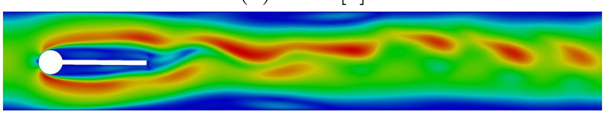
FIGURE 1.11 – Comparaison entre les taux de décroissance lorsque $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ et $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, dans le cas où le coefficient de rigidité $\boldsymbol{\alpha = 10^{-1}}$ et l'amplitude de perturbation $\beta = 1.5$.



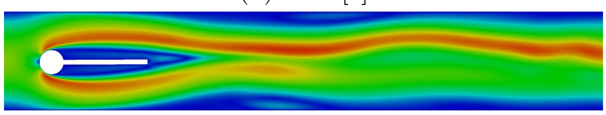
(a) $t = 1[s]$



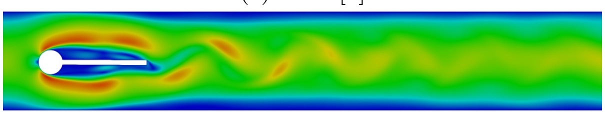
(b) $t = 1[s]$



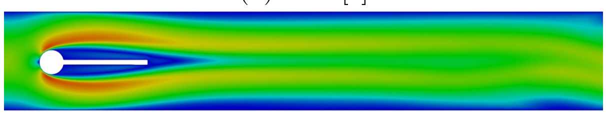
(c) $t = 2[s]$



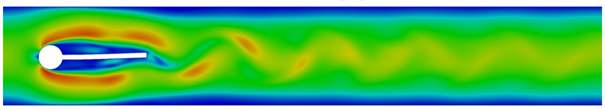
(d) $t = 2[s]$



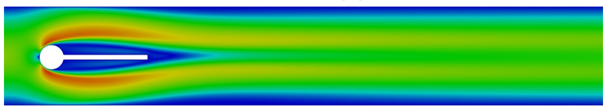
(e) $t = 4[s]$



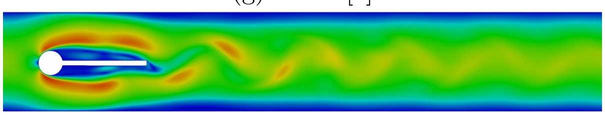
(f) $t = 4[s]$



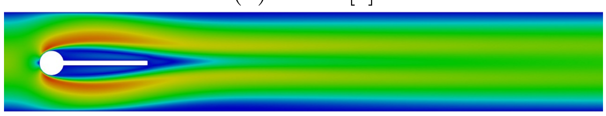
(g) $t = 11[s]$



(h) $t = 11[s]$



(i) $t = 15[s]$



(j) $t = 15[s]$

FIGURE 1.12 – Snapshots du module de la vitesse du fluide à différents instants, correspondant au coefficient de rigidité $\boldsymbol{\alpha = 10^{-1}}$, à l'amplitude de perturbation $\beta = 1.5$, et au sous-espace instable $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$. Dans la colonne de gauche (a)-(c)-(e)-(g)-(i), nous montrons le module de la vitesse du fluide pour le cas sans contrôle, tandis que dans la colonne de droite (b)-(d)-(f)-(h)-(j), nous montrons la vitesse du fluide lorsque le contrôle est appliqué.

3. **Influence de l'amplitude de perturbation**.

Comme pour les deux expériences précédentes, en fixant $Re = 200$ et $\gamma = 10^{-6}$, nous faisons varier le paramètre α et l'amplitude β de la perturbation. Par exemple, pour $\alpha = 10^{-1}$, nous montrons dans la Figure 1.13 l'évolution des normes $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$ et $\|f\|_{L^2}$, pour trois valeurs différents de l'amplitude β et les contrôles basés sur les sous-espaces instables $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ et l'autre sur $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$.

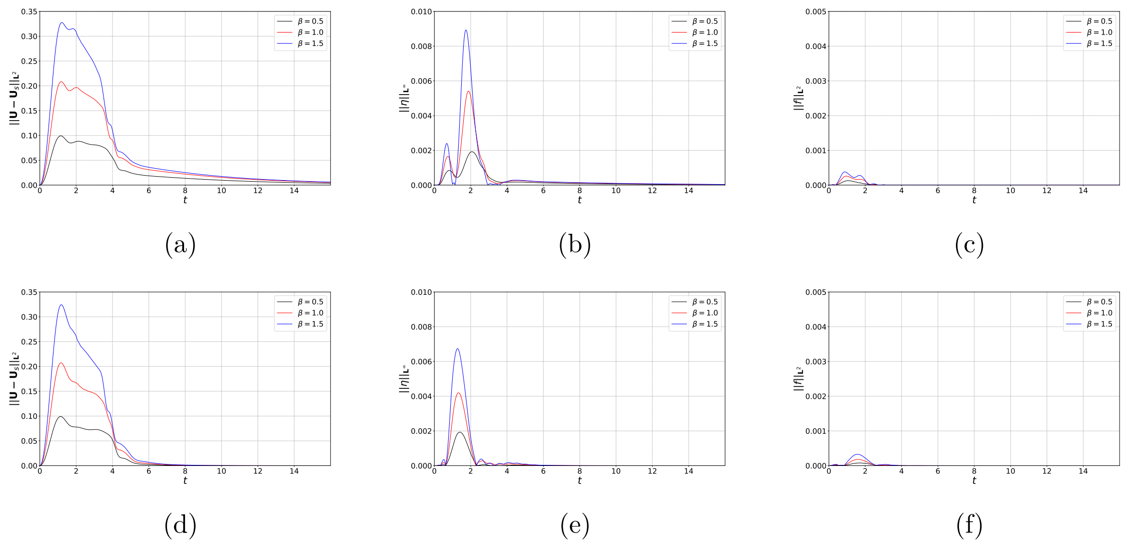


FIGURE 1.13 – Évolution de $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, (b) $\|\eta\|_{L^\infty}$ et (c) $\|f\|_{L^2}$, pour trois valeurs différentes du paramètre β. Ligne (a)-(b)-(c) : Contrôle basé sur $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$. Ligne (d)-(e)-(f) : Contrôle basé sur $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$.

1.4 Perspectives

Ci-dessous, nous détaillons quelques perspectives.

– *Modélisation*

Tout d'abord, comme expliqué dans l'annexe A, dans la modélisation de la dynamique de la structure, nous supposons des conditions homogènes sur l'extrémité droite de la poutre, ce qui implique une simplification du modèle. Cependant, il serait intéressant d'envisager une condition non homogène à l'extrémité droite de la poutre qui tienne compte du terme résultant de l'identité énergétique due à la contribution latérale sur la poutre. Étant donnée qu'il s'agit d'une condition non standard, il est important de comprendre d'abord le comportement de la dynamique de la structure de manière isolée.

Un autre élément intéressant à inclure dans un modèle plus réaliste consiste à permettre à la poutre de se déplacer latéralement en plus de transverse. Cela impliquerait d'inclure une équation supplementaire qui tienne compte de la dynamique du mouvement latéral. Il faudrait alors préciser à la fois les conditions aux limites et les conditions cinématique et dynamique. En principe, il n'est pas clair que les outils utilisés au chapitre 3 pour étudier l'existence de solutions fortes soient directement adaptables à un tel contexte. D'autre part, une mise en œvre de la strategie utilisée au chapitre 4 pour réaliser les simulations numériques semble adaptable.

– *Simulations numériques*

  En ce qui concerne la stratégie utilisé tant dans la mise en œuvre numérique du problème direct au chapitre 4 que dans le traitement du problème de stabilisation au chapitre 6, il serait intéressant de mener une analyse numérique d'estimation d'erreur.

  D'autre part, la construction de la transformation ALE utilisée pour traiter le problème direct au chapitre 3 est basée sur l'extension harmonique de la trace de la vitesse du fluide. Comme indiqué dans [Wic11], ce choix particulier est efficace pour les petites déformations de la structure. Une première étape pour contourner cette difficulté consiste à mettre en œvre des stratégies telles que celles proposées dans [Wic11] et [HC23], par exemple.

– *Estimations de modèles*

  La stratégie développée au chapitre 5 pour l'étude du problème de stabilisation au niveau théorique, ainsi que les simulations numériques dévéloppées au chapitre 6, ont été réalisées en partant de l'hypothèse que l'état du système est connu à chaque instant. En pratique, cette hypothèse n'est pas réaliste. Une altérnative à cette dernière consiste à utiliser une estimation de l'état, basée par exemple sur des mesures de pression généralement sur le bord de la structure. Avant de s'intéresser à une telle question pour le modèle considéré dans ce travail, une première étape pourrait consister à étudier un modèle simplifié pour la structure.

Chapter 2

Study of the stationary Stokes system with mixed boundary conditions in non-convex curvilinear polygonal domains

Abstract of the current chapter

In this chapter,we consider the stationary Stokes system with mixed boundary conditions, of Dirichlet and Neumann types, in a bounded non-convex curvilinear polygonal domain of $\mathbb{R}^2$. We prove, in particular, a precise regularity result in heterogeneous Sobolev spaces taking into account the fact that the expected regularity is of different nature near the corners of the domain and near the Dirichlet-Neumann transition points. Then, we prove the analyticity of the semigroup generated by the Stokes operator in an appropriate functional setting. We also give a characterization of the stationary Stokes system as an operator equation.

Outline of the current chapter

2.1 Introduction **39**

2.1.1 Statement of the problem 39

2.1.2 Motivation 40

2.1.3 Outline 41

2.2 Classes of domains and functional setting **41**

2.2.1 Classes of domains 41

2.2.2 Functional setting 42

2.3 Existence, uniqueness and regularity **45**

2.4 Rewriting the Stokes system as an operator equation **47**

2.4.1 The Leray projector 47

2.4.2 Stokes operator and the analyticity of its underlying semigroup . . 51

2.4.3 Expression of the pressure 58

2.4.4 System reformulated as an operator equation 61

Appendix A : Study of the Stokes resolvent system **62**

Appendix B : Technical result adapted from Pazy's book **64**

Appendix C : Justification of the identity (2.4.55) (formal computations) **65**

2.1 Introduction

2.1.1 Statement of the problem

We are interested in studying the stationary Stokes equation in a two dimensional curvilinear polygonal and Lipschitz domain Ω with boundary $\Gamma = \Gamma_d \cup \Gamma_n$, where Dirichlet boundary conditions are applied on Γ_d and homogeneous Neumann boundary conditions are prescribed on Γ_n. The precise assumptions on Ω and Γ are given in Subsection 2.2.1.

We consider the system

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{w},\pi) = \mathbf{F} \ \text{ in } \Omega, \ \ \operatorname{div}\mathbf{w} = h \ \text{ in } \Omega, \\ \mathbf{w} = \mathbf{g} \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \end{cases} \tag{2.1.1}$$

where $\mathbf{F}, h$ and $\mathbf{g}$ are given data and the Cauchy stress tensor $\sigma(\mathbf{w},\pi)$ is given by

$$\sigma(\mathbf{w},\pi) = 2\nu\varepsilon(\mathbf{w}) - \pi I, \quad \varepsilon(\mathbf{w}) = \frac{1}{2}(\nabla\mathbf{w} + (\nabla\mathbf{w})^\top),$$

with $\nu > 0$ denoting the viscosity of the fluid. In order to present the main contribution of the present paper, let us recall two regularity results concerning Stokes systems. From [MR10, Theorem 9.4.5] we know that the solution $(\mathbf{w},\pi)$ to the Stokes system (2.1.1) with mixed Dirichlet-Neumann boundary conditions satisfies

$$\begin{aligned} &(\mathbf{w},\pi) \text{ belongs to the weighted Sobolev space } \mathbf{H}^2_\delta(\Omega) \times H^1_\delta(\Omega) \\ &\text{when } \mathbf{F} \in \mathbf{L}^2_\delta(\Omega), \ h \in H^1_\delta(\Omega) \text{ and } \mathbf{g} \in \mathbf{H}^{\frac{3}{2}}_\delta(\Gamma_d), \end{aligned} \tag{R1}$$

for all $\delta \in (\delta^*, 1)$, where $\delta^* \in (0, 1/2)$. The weighted Sobolev spaces $\mathbf{H}^2_\delta(\Omega)$, $H^1_\delta(\Omega)$, and some others are introduced in Section 2.2. Some other results concerning systems of the form (2.1.1) may be found in [BGM10, Proposition A.1].

On the other hand, for the Stokes system with only a Dirichlet boundary condition, $\mathbf{w} = \mathbf{g}$ on $\Gamma = \Gamma_d$ and $\Gamma_n = \emptyset$, under a suitable compatibility condition on h and $\mathbf{g}$, we have

$$\begin{aligned} &(\mathbf{w},\pi) \text{ belongs to the Sobolev space } \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \times H^{\frac{1}{2}+\alpha}(\Omega), \\ &\text{when } \mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\ h \in H^{\frac{1}{2}+\alpha}(\Omega),\ \mathbf{g} \in \mathbf{H}^{1+\alpha}(\Gamma) \text{ for all } \alpha \in (0,\alpha^*), \end{aligned} \tag{R2}$$

where the critical exponent $\alpha^* \in (0, 1/2)$ depends on the angles at the corners of Γ.

This result is proved in [Dau89, Theorem 5.5(a)] when $\mathbf{g} = 0$ and extended to $\mathbf{g} \neq 0$ in [BR, Corollary 3.3].

One of the objectives of this article is to study the existence, uniqueness and regularity of solutions to system (2.1.1) in heterogeneous Sobolev spaces. Here, the heterogeneous spaces introduced in (2.2.7) are used to characterize functions having different regularity in distinct zones of the domain Ω. In particular, we would like to recover simultaneously the regularity in weighted Sobolev spaces, as in (R1), in a neighborhood of Γ_n, and to recover the fractional Sobolev regularity, as in (R2), in a neighborhood of corners of Γ_d corresponding to Dirichlet boundary conditions on both sides of the corner.

The main contributions of this article are outlined below.

— In Theorem 2.3.2, we show that if $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)$, $h \in H^{\frac{1}{2}+\alpha}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$ (with $\alpha \in (0,\alpha^*)$), and if, in a neighborhood of Γ_n, $\mathbf{F}$ and h belong to $\mathbf{L}^2$ and H^1, respectively, then the solution $(\mathbf{w},\pi)$ to system (2.1.1) belongs to $\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \times H^{\frac{1}{2}+\alpha}(\Omega)$, and in $\mathbf{H}^2_\delta \times H^1_\delta$ in a neighborhood of Γ_n (with $\delta \in (\delta^*, 1)$). The precise result is stated in Theorem 2.3.2 in terms of the heterogeneous Sobolev spaces introduced in Section 2.2, see (2.2.7). The main idea of the proof is to combine the regularity results (R1) and (R2) employing a truncation argument.

— One motivation for studying system (2.1.1) is to provide results needed to study a fluid-structure interaction system investigated by the authors in Chapter 3. In that respect, the study of the linear parabolic system

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \operatorname{div} \sigma(\mathbf{w}, \pi) = \mathbf{F} \ \text{ in } (0,T) \times \Omega, \ \operatorname{div} \mathbf{w} = h \ \text{ in } (0,T) \times \Omega, \\ \mathbf{w} = \mathbf{g} \ \text{ on } (0,T) \times \Gamma_d, \ \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \ \text{ on } (0,T) \times \Gamma_n, \\ \mathbf{w}(0) = \mathbf{w}_0 \ \text{ in } \Omega, \end{cases} \tag{2.1.2}$$

with optimal regularity of the solutions is a crucial step. For this analysis, in addition to the regularity in the heterogeneous spaces mentionned above, we need the analyticity of the semigroup generated by the Stokes operator $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$. The precise result is stated in Theorem 2.4.2. The proof of this result relies on Proposition 2.4.7, which in turn relies on the existence of an extension operator $E : \Omega \longrightarrow \widetilde{\Omega}$, preserving both the divergence-free condition and the trace on Γ_n. Here, $\widetilde{\Omega}$ is the orthogonal symmetrization of the domain Ω with respect to Γ_n.
With this property in hand, we are able to split the system (2.1.1) (or (2.1.2)) into three parts: an operator equation for $P\mathbf{w}$ (where P is the Leray projector in this functional framework), an algebraic equation for $(I - P)\mathbf{w}$, and a precise characterization of the pressure π. This is given in Theorem 2.4.3.
We have to emphasize here that the existence and the regularity of π, obtained in Theorem 2.3.2, is not sufficient to deal with fluid-structure-interaction systems as those studied in Chapter 3. Indeed, the splitting of π introduced in Theorem 2.4.3 plays a crucial role to characterize the so-called *added mass operator* of the fluid-structure-interaction system that we consider.

— The starting point for proving the two aforementioned results is the characterization of the space $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$ (the dual of the space $\mathbf{V}^{s}_{n,\Gamma_d}(\Omega)$ introduced in (2.4.24)) when $0 < s < 1/2$. To the best of our knowledge, the result established in Lemma 2.4.5 has not been previously reported in the literature. We emphasize that this characterization is based on regularity results for the solution of a suitable elliptic equation with mixed boundary conditions, see Subsection 2.4.1.

2.1.2 Motivation

Let us now explain why, in the fluid-structure interaction system investigated in Chapter 3, we need to introduce the functional framework of heterogeneous Sobolev spaces.

The fluid-structure system considered in Chapter 3 models the interaction between the incompressible Navier-Stokes equations in a 2D rectangular domain with mixed boundary conditions, and an elastic structure governed by the Euler-Bernoulli equation with a clamped boundary condition at one extremity of the elastic beam and a Neumann boundary condition (a type of *free boundary condition*) at the other extremity. Figure 2.1 shows the corresponding geometric configuration.

A similar fluid-structure interaction system has been studied in [FNR19], but with clamped boundary conditions at the extremities of the elastic beam. In that case the nonlinear problem can be studied in the framework of the weighted Sobolev spaces $\mathbf{H}^2_\delta(\Omega)$ and $H^1_\delta(\Omega)$, introduced above.

In the case where a Neumann boundary condition is prescribed at one extremity of the elastic beam, the linearized fluid-structure interaction system can still be studied in the framework of the weighted Sobolev spaces $\mathbf{H}^2_\delta(\Omega)$ and $H^1_\delta(\Omega)$. But this approach is not sufficient to deal with the nonlinear fluid-structure interaction system with a Neumann boundary condition prescribed at one extremity of the elastic beam. This is due to a mismatch between the regularity results, obtained in the weighted Sobolev spaces, for solutions to nonhomogeneous linear models and the corresponding regularity of some nonlinear terms of the model (which will play the role of the nonhomogeneous terms in a fixed point procedure). This is why the results presented in the work are novel and essential contributions to deal with certain nonlinear fluid-structure interaction systems.

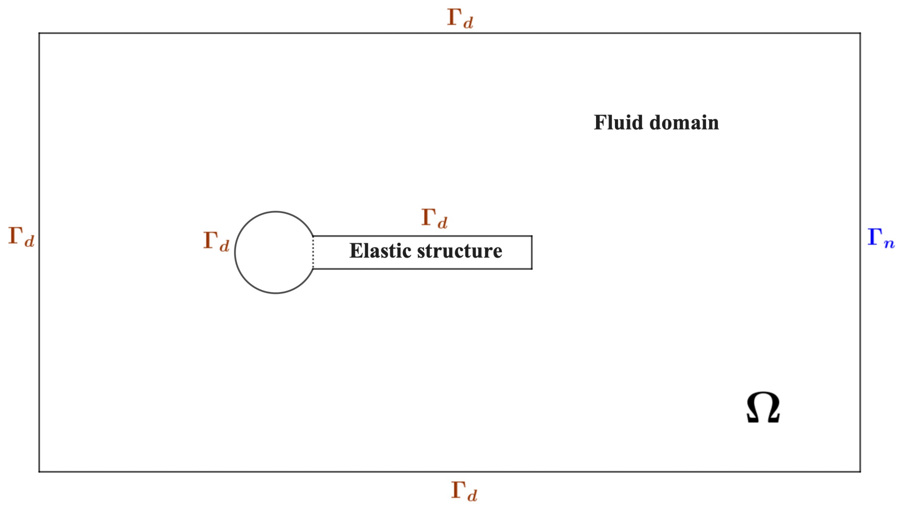


Figure 2.1 – Geometrical configuration used in the fluid-structure problem studied in Chapter 3.

2.1.3 Outline

The chapter is organized as follows. In Section 2.2, we introduce the classes of domains and the notation used throughout the paper. In Section 2.3, we establish a regularity result of system (2.1.1) in heterogeneous Sobolev spaces. Section 2.4 is divided in four subsections. In Subsection 2.4.1, we present results concerning the Leray projector (see Corollary 2.4.1), which are based on a careful analysis of a certain elliptic equation. Then, in Subsection 2.4.2, we prove, the analyticity of the underlying semigroup associated to the Stokes operator on $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$. Next, in Subsection 2.4.3, we present a useful representation of the pressure π of system (2.1.1). Finally, in Subsection 2.4.4, we provide a characterization of system (2.1.1) as an operator equation.

2.2 Classes of domains and functional setting

2.2.1 Classes of domains

We assume that Ω is a Lipschitz curvilinear polygonal domain in $\mathbb{R}^2$. More precisely, it satisfies the following geometric hypotheses:

- $(H1)$ Ω is bounded and connected.
- $(H2)$ The boundary Γ of Ω is given by $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$, where each arc Γ_i is assumed to be smooth. We also assume that for all $i, j \in \{1, \ldots, N\}$, with $i \neq j$, we have either $\Gamma_i \cap \Gamma_j = \emptyset$ or $\Gamma_i \cap \Gamma_j = \{J_{i,j}\}$, where $J_{i,j}$ denotes the vertex joining the arcs Γ_i and Γ_j. We will denote by $\mathcal{J}$ the set of vertices of Γ.
- $(H3)$ At the neighborhood of any vertex $J_{i,j} \in \mathcal{J}$, Ω is locally diffeomorphic to a neighborhood of zero in a plane sector of angle $\theta_{i,j}$ in $(0, 2\pi)$.

We now state assumptions on Γ_d and Γ_n, where Dirichlet boundary conditions are applied on Γ_d and homogeneous Neumann boundary conditions on Γ_n. See Figure 2.2.

- $(H4)$ We assume that there exists $j_0 \in \{1, \ldots, N\}$ such that $\Gamma_n = \Gamma_{j_0}$, where Γ_n is a segment with extremities A and E. We also assume that there exist two lines ℓ_1 and ℓ_2 passing through A and E, respectively, such that $\ell_1 \cap \overline{\Omega} = \{A\}$ and $\ell_2 \cap \overline{\Omega} = \{E\}$. In particular, if D_n denotes the line containing Γ_n, then the domain Ω is contained in one of the half-planes supported by D_n.

Remark 1. Since $\Gamma_n = \Gamma_{j_0}$, we notice that we have $\Gamma_d = \bigcup_{i\in\{1,\dots,N\},\ i\neq j_0} \Gamma_i$. The results of the paper can be generalized without difficulty to the case where Γ_n is a union of several segments Γ_i with $i \in \{1,\dots,N\}$, without junction between two Neumann boundary conditions.

We use the fact that Γ_n is a segment only to simplify the analysis of the Poisson equation with mixed boundary conditions (see Proposition 2.4.2 and Lemma 2.4.1), but the results can be easily extended to the case where Γ_n is a smooth curve, as in [MRR20, Proof of Theorem 4.2].

Notice that regularity results similar to those of Lemma 2.4.1 are stated in [Dau88, Chapter 8, Corollary 23.5, and Remark 23.6, page 197] in the case where Γ_n is a regular curve and the angle condition stated in $(H4)$ is satisfied. The assumption $(H4)$ is also used in the proof of Lemma 2.4.7.

Remark 2. The last constraint imposed in (H4) simplifies the analysis we develop in Section 2.4 which involves deducing the regularity of solution to the Poisson equation with mixed Dirichlet-Neumann boundary conditions from the regularity of the solution to the Poisson equation with only Neumann boundary conditions, but in a larger domain obtained by symmetry.

When this geometric constraint is not satisfied, it is necessary to use an additional truncation argument in order to avoid the self-intersection of Ω with its symmetric. We do not explicit this technical step in this work.

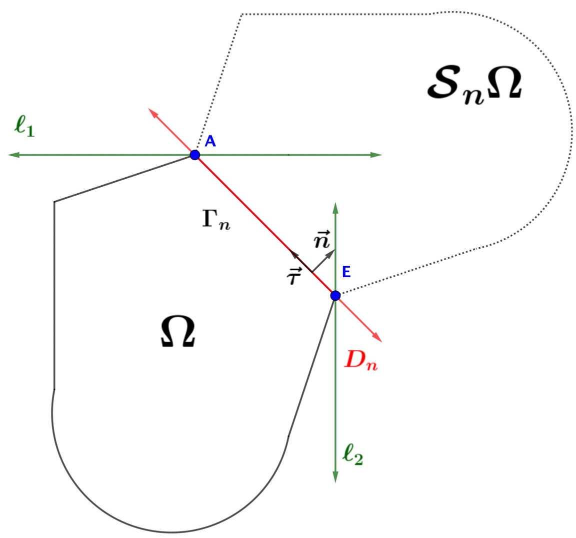


Figure 2.2 – Example symmetrization.

2.2.2 Functional setting

Usual and weighted Sobolev spaces

We first set $\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2)$ and, for $s > 0$, $\mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2)$. We also introduce the following functional spaces:

$$\mathbf{H}^1_{\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \mathbf{u} = 0 \ \text{ on } \Gamma_d\},$$
$$\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_d\},$$
$$\mathbf{V}^1_{\Gamma_d}(\Omega) = \mathbf{H}^1_{\Gamma_d}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega).$$

The dual of $\mathbf{H}^1_{\Gamma_d}(\Omega)$, with $\mathbf{L}^2(\Omega)$ as a pivot space, is denoted by $\mathbf{H}^{-1}_{\Gamma_d}(\Omega)$. We denote by $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ the dual of $\mathbf{V}^1_{\Gamma_d}(\Omega)$ with $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ as pivot space. We have

$$\mathbf{V}^1_{\Gamma_d}(\Omega) \hookrightarrow \mathbf{V}^0_{n,\Gamma_d}(\Omega) \hookrightarrow \mathbf{V}^{-1}_{\Gamma_d}(\Omega)$$

with dense and continuous embeddings. For $0 < s < 1/2$, we introduce the spaces

$$\mathbf{H}^s_{\Gamma_d}(\Omega) = [\mathbf{L}^2(\Omega), \mathbf{H}^1_{\Gamma_d}(\Omega)]_s,$$

equipped with the classical Sobolev norm $\|\cdot\|_{\mathbf{H}^s}$. The dual of $\mathbf{H}^s_{\Gamma_d}(\Omega)$ is denoted by $\mathbf{H}^{-s}_{\Gamma_d}(\Omega)$. For the definition of interpolation spaces using the complex method we refer to [Tar07], [Tri78].

For $s > 0$, $\mathcal{H}^s(\Omega) = H^s(\Omega) \cap L^2_0(\Omega)$, where $L^2_0(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega f = 0\}$. The dual of $\mathcal{H}^s(\Omega)$, with $L^2_0(\Omega)$ as a pivot, is denoted by $\mathcal{H}^{-s}(\Omega)$.

We now introduce weighted Sobolev spaces as in [MR10]. For $\beta > 0$, we introduce the norms

$$\begin{aligned} \|\mathbf{w}\|_{\mathbf{H}^2_\beta} &:= \left(\sum_{|k|=0}^{2} \sum_{i=1}^{2} \int_\Omega \Big(\prod_{J\in\mathcal{J}} r_J^{2\beta} \Big) |\partial^k w_i|^2 dx \right)^{1/2}, \quad \mathbf{w} \in C^\infty(\overline{\Omega}; \mathbb{R}^2) \\ \|p\|_{H^1_\beta} &:= \left(\sum_{|k|=0}^{1} \int_\Omega \Big(\prod_{J\in\mathcal{J}} r_J^{2\beta} \Big) |\partial^k p|^2 dx \right)^{1/2}, \quad p \in C^\infty(\overline{\Omega}; \mathbb{R}) \end{aligned} \tag{2.2.1}$$

where r_J stands for the distance to the junction point $J \in \mathcal{J}$, $k = (k_1, k_2) \in \mathbb{N}^2$ denotes a two-index with length $|k| = k_1 + k_2$, ∂^k denotes the corresponding partial differential operator and $\mathbf{w} = (w_1, w_2)$. We denote by $\mathbf{H}^2_\beta(\Omega; \mathbb{R}^2)$ (respectively, $H^1_\beta(\Omega)$) the closure of $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ (respectively, $C^\infty(\overline{\Omega})$) in the norm $\|\cdot\|_{\mathbf{H}^2_\beta}$ (respectively, $\|\cdot\|_{H^1_\beta}$).

Exponent of weighted Sobolev spaces

We need to introduce an exponent $\delta^* \in (0, 1/2)$ related to the presence of mixed Dirichlet-Neumann boundary conditions for the Stokes problem.

• Due to [MR10, Theorem 9.4.5], there exists $\delta^* \in (0, 1/2)$ associated to the Stokes system in Ω with mixed Dirichlet/Neumann boundary conditions

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{u}, p) = \mathbf{F} \;\text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = h \;\text{ in } \Omega, \\ \mathbf{u} = \mathbf{g} \;\text{ on } \Gamma_d, \;\; \sigma(\mathbf{u}, p)\mathbf{n} = 0 \;\text{ on } \Gamma_n, \end{cases} \tag{2.2.2}$$

such that $(\mathbf{u}, p)$ obeys

$$\|\mathbf{u}\|_{\mathbf{H}^2_\delta} + \|p\|_{H^1_\delta} \leq C_\delta \left(\|\mathbf{F}\|_{\mathbf{L}^2} + \|h\|_{H^1} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_d)} \right), \tag{2.2.3}$$

for all $\delta \in (\delta^*, 1)$.

Remark 3. Notice that the result from [MR10] that we mention above is stated for a polygonal domain and not for a curvilinear polygonal domain. However using a diffeomorphism and a localization argument as in [MRR20, Proof of Theorem 4.2], the results from [MR10], that we need here, can be extended to the case of curvilinear polygonal domains.

Regularity exponents

We now need to introduce a regularity index $\alpha^* \in (0, 1/2)$, which will be used throughout the article. This index depends on the angles at the corners of Γ and on the type of boundary conditions prescribed on both sides of the corners. It allows us to state regularity results for the Stokes and Poisson problems in Ω and in $\widetilde{\Omega}$ defined by

$$\widetilde{\Omega} = \Omega \cup \Gamma_n \cup \mathcal{S}_n \Omega, \tag{2.2.4}$$

where $\mathcal{S}_n$ is the orthogonal symmetry with respect to D_n, where D_n is the straight line in $\mathbb{R}^2$ containing Γ_n (see Figure 2.2).

We set

$$\alpha^* := \min\{\alpha_{SD}, \alpha_{SM}, \alpha_{PN}\}, \tag{2.2.5}$$

where the parameters α_{SD}, α_{SM}, and α_{PN} are defined below:

• From [Dau89, Theorem 5.5(a)] we know that there exists $\alpha_{SD} \in (0, 1/2)$ associated to the Stokes equation in Ω with homogeneous Dirichlet boundary conditions

$$-\operatorname{div} \sigma(\mathbf{u}, p) = \mathbf{F} \ \text{ in } \Omega, \ \ \operatorname{div} \mathbf{u} = h \ \text{ in } \Omega, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma,$$

such that $(\mathbf{u}, p)$ satisfies

$$\|\mathbf{u}\|_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|p\|_{H^{\frac{1}{2}+\alpha}} \leq C_\alpha \Big(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} + \|h\|_{\mathcal{H}^{\frac{1}{2}+\alpha}} \Big),$$

for all $\alpha \in (0, \alpha_{SD})$.

• We denote by $\alpha_{SM} \in (0, 1/2)$ the regularity index associated to the Stokes system in Ω with mixed Dirichlet/Neumann boundary conditions (2.2.2). For all $\delta \in (\delta^*, 1)$, from [MR10, Lemma 6.2.1] we have the continuous embeddings $\mathbf{H}^2_\delta(\Omega) \hookrightarrow \mathbf{H}^{2-\delta}(\Omega)$ and $H^1_\delta(\Omega) \hookrightarrow H^{1-\delta}(\Omega)$. Thus, we set $\alpha_{SM} := 1/2 - \delta^*$.

• We denote by $\alpha_{PN} \in (0, 1/2)$ the regularity index associated to the Poisson problem in $\widetilde{\Omega}$ with homogeneous Neumann boundary conditions

$$-\Delta \varphi = \zeta \ \text{ in } \widetilde{\Omega}, \ \ \frac{\partial \varphi}{\partial \mathbf{n}} = 0 \ \text{ on } \widetilde{\Gamma} = \partial \widetilde{\Omega}.$$

The parameter α_{PN} is introduced in [Dau88, Chapter 8, Corollary 23.5, page 197] and is such that for all $\alpha \in (0, \alpha_{PN})$, we have

$$\|\varphi\|_{\mathcal{H}^{\frac{3}{2}+\alpha}(\widetilde{\Omega})} \leq C_\alpha \|\zeta\|_{\mathcal{H}^{-\frac{1}{2}+\alpha}(\widetilde{\Omega})}.$$

($\mathcal{H}^{-\frac{1}{2}+\alpha}(\widetilde{\Omega})$ is defined similarly to $\mathcal{H}^{-\frac{1}{2}+\alpha}(\Omega)$.)

Heterogeneous Sobolev spaces

Let us denote by $\mathcal{J}_{dd} \subset \mathcal{J}$ the set of corner vertices corresponding to a junction between two Dirichlet boundary conditions. Let $\mathcal{U}$ and $\mathcal{V}$ be two disjoint open subsets of $\mathbb{R}^2$ such that $\mathcal{J}_{d,d} \subset \mathcal{U}$ and $\Gamma_n \subset \mathcal{V}$. In particular, $\mathcal{V}$ does not countain any corner with a Dirichlet-Dirichlet junction. Let us now introduce any cut-off function $\Psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $0 \leq \Psi \leq 1$ and

$$\Psi = 1 \text{ in } \mathcal{U} \text{ and } \Psi = 0 \text{ in } \mathcal{V}. \tag{2.2.6}$$

Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. We introduce the following heterogeneous Sobolev spaces

$$\begin{aligned}
\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) &= \left\{ \mathbf{F} \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1(\Omega) \right\}, \\
H_\delta^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H_\delta^1(\Omega) \right\}, \\
\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) &= \left\{ \mathbf{v} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{v} \in \mathbf{H}_\delta^2(\Omega) \right\},
\end{aligned} \tag{2.2.7}$$

which are respectively endowed with the norms

$$\begin{aligned}
\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}} &:= \left(\|\mathbf{F}\|^2_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}} + \|(1-\Psi)\mathbf{F}\|^2_{\mathbf{L}^2} \right)^{1/2}, \\
\|p\|_{H^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H^1} \right)^{1/2}, \\
\|p\|_{H_\delta^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H_\delta^1} \right)^{1/2}, \\
\|\mathbf{u}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}} &:= \left(\|\mathbf{u}\|^2_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|(1-\Psi)\mathbf{u}\|^2_{\mathbf{H}_\delta^2} \right)^{1/2}.
\end{aligned}$$

Notice that those spaces actually depend on the choice of the cut-off function Ψ, but this choice will be fixed all along this chapter. That is the reason why, for simplicity, we do not explicitly mention Ψ in the notation we propose for those spaces.

2.3 Existence, uniqueness and regularity

Let us assume that $\mathbf{F} \in \mathbf{H}_{\Gamma_d}^{-1}(\Omega)$, $h \in L^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_d)$. We will say that the pair $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ is a variational solution of the system (2.1.1) if and only if it satisfies the following mixed variational formulation:

$$\begin{cases}
a(\mathbf{w}, \boldsymbol{\phi}) - b(\boldsymbol{\phi}, \pi) = \langle \mathbf{F}, \boldsymbol{\phi} \rangle_{\mathbf{H}_{\Gamma_d}^{-1}, \mathbf{H}_{\Gamma_d}^1} \text{ for all } \boldsymbol{\phi} \in \mathbf{H}_{\Gamma_d}^1(\Omega), \\
b(\mathbf{w}, \psi) = \displaystyle\int_\Omega h\psi \text{ for all } \psi \in L^2(\Omega), \\
\mathbf{w} = \mathbf{g} \text{ on } \Gamma_d,
\end{cases} \tag{2.3.1}$$

where

$$a(\mathbf{w}, \phi) = 2\nu \int_\Omega \varepsilon(\mathbf{w}) : \varepsilon(\boldsymbol{\phi}) \text{ and } b(\mathbf{w}, \psi) = \int_\Omega (\operatorname{div} \mathbf{w})\, \psi.$$

Such a solution exists and is unique as stated in the following result.

Theorem 2.3.1. *Let us assume that $(\mathbf{F}, h) \in \mathbf{H}_{\Gamma_d}^{-1}(\Omega) \times L^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_d)$. System (2.1.1) admits a unique variational solution $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$.*

Proof. Since $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_d)$, there exists a lifting $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v} = \mathbf{g}$ on Γ_d, satisfying the estimate $\|\mathbf{v}\|_{\mathbf{H}^1} \leq C\|\mathbf{g}\|_{\mathbf{H}^{1/2}}$. Then, we look for the solution $\mathbf{w}$ to (2.1.1) in the form $\mathbf{w} = \widetilde{\mathbf{w}} + \mathbf{v}$. We notice that the couple $(\widetilde{\mathbf{w}}, \pi)$ satisfies

$$\begin{cases}
a(\widetilde{\mathbf{w}}, \boldsymbol{\phi}) - b(\boldsymbol{\phi}, \pi) = \langle \mathbf{F}, \boldsymbol{\phi} \rangle_{\mathbf{H}_{\Gamma_d}^{-1}, \mathbf{H}_{\Gamma_d}^1} - a(\mathbf{v}, \boldsymbol{\phi}) \text{ for all } \boldsymbol{\phi} \in \mathbf{H}_{\Gamma_d}^1(\Omega), \\
b(\widetilde{\mathbf{w}}, \psi) = \displaystyle\int_\Omega h\psi - b(\mathbf{v}, \psi) \text{ for all } \psi \in L^2(\Omega), \\
\widetilde{\mathbf{w}} = 0 \text{ on } \Gamma_d.
\end{cases} \tag{2.3.2}$$

The existence of a unique solution $(\widetilde{\mathbf{w}}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ to the system (2.3.2) follows from the Babuška-Brezzi Theorem [GR86, Theorem 4.1, Chapter I]. For the proof of the surjectivity of the operator $\operatorname{div}(\cdot) \in \mathcal{L}(\mathbf{H}^1_{\Gamma_d}(\Omega), L^2(\Omega))$ we refer the reader to [EG21, Lemma 53.9, p. 409]. $\square$

We now establish a regularity result in heterogeneous Sobolev spaces for the solution of system (2.1.1).

Theorem 2.3.2. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Assume that $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $h \in H^{\frac{1}{2}+\alpha,1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$. Then, the variational solution $(\mathbf{w}, \pi)$ of system (2.1.1) belongs to $\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$. Moreover, there exists a constant $C_{\alpha,\delta} > 0$ such that*

$$\|\mathbf{w}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)} + \|\pi\|_{H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)} \leq C_{\alpha,\delta} \left(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} + \|h\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_d)} \right). \tag{2.3.3}$$

Proof. Let $(\mathbf{w}, \pi) \in \mathbf{H}^1_\Gamma(\Omega) \times L^2(\Omega)$ be the solution to (2.1.1) whose existence is shown in Theorem 2.3.1. Let us consider the cut-off function Ψ defined in (2.2.6). Then, the pair $(\mathbf{w}_1, \pi_1) = (\Psi\mathbf{w}, \Psi\pi)$ satisfies

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{w}_1, \pi_1) = \mathbf{F}_1 \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{w}_1 = h_1 \ \text{ in } \Omega, \\ \mathbf{w}_1 = \Psi\mathbf{g} \ \text{ on } \Gamma_d, \quad \mathbf{w}_1 = 0 \ \text{ on } \Gamma_n, \end{cases} \tag{2.3.4}$$

where

$$\mathbf{F}_1 = \Psi\mathbf{F} + \pi\nabla\Psi - \nu\left(\mathbf{w}\Delta\Psi + 2(\nabla\Psi \cdot \nabla)\mathbf{w}\right) - \nu(\operatorname{div}\mathbf{w}\nabla\Psi + \nabla(\mathbf{w}\cdot\nabla\Psi))$$

and

$$h_1 = \Psi h + \nabla\Psi \cdot \mathbf{w}.$$

Since $(\mathbf{w}, \pi) \in \mathbf{H}^1_\Gamma(\Omega) \times L^2(\Omega)$, we have that $\mathbf{F}_1 \in \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)$ and $h_1 \in H^{\frac{1}{2}+\alpha}(\Omega)$, with $0 < \alpha < \alpha^*$. Then, it follows from [Dau89, Theorem 5.5(a)] and [BR, Theorem 3.2 and Corollary 3.3] that

$$\|\mathbf{w}_1\|_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|\pi_1\|_{H^{\frac{1}{2}+\alpha}} \leq C_\alpha \left(\|\mathbf{F}\|_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}} + \|h\|_{H^{\frac{1}{2}+\alpha}} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}} \right). \tag{2.3.5}$$

On the other hand, we notice that the pair $(\mathbf{w}_2, \pi_2) = ((1-\Psi)\mathbf{w}, (1-\Psi)\pi)$ satisfies:

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{w}_2, \pi_2) = \mathbf{F}_2, \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{w}_2 = h_2 \ \text{ in } \Omega, \\ \mathbf{w}_2 = (1-\Psi)\mathbf{g} \ \text{ on } \Gamma_d, \\ \sigma(\mathbf{w}_2, \pi_2)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \end{cases} \tag{2.3.6}$$

where

$$\mathbf{F}_2 = (1-\Psi)\mathbf{F} - \pi\nabla\Psi + \nu\left(\mathbf{w}\Delta\Psi + 2(\nabla\Psi \cdot \nabla)\mathbf{w}\right) + \nu(\operatorname{div}\mathbf{w}\nabla\Psi + \nabla(\mathbf{w}\cdot\nabla\Psi))$$

and

$$h_2 = (1-\Psi)h - \mathbf{w}\cdot\nabla\Psi.$$

Since $(1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $(1-\Psi)h \in H^1(\Omega)$, we deduce that $\mathbf{F}_2 \in \mathbf{L}^2(\Omega)$ and $h_2 \in H^1(\Omega)$.

Then, from [MR10, Theorem 9.4.5], we deduce that $(\mathbf{w}_2, \pi_2) \in \mathbf{H}^2_\delta(\Omega) \times H^1_\delta(\Omega)$ and

$$\begin{aligned}&\|\mathbf{w}_2\|_{\mathbf{H}^2_\delta} + \|\pi_2\|_{H^1_\delta}\\ &\leq C\Big(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} + \|h\|_{H^{\frac{1}{2}+\alpha}} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}} + \|(1-\Psi)\mathbf{F}\|_{\mathbf{L}^2} + \|(1-\Psi)h\|_{H^1}\Big).\end{aligned} \tag{2.3.7}$$

Moreover, thanks to [MR10, Lemma 6.2.1], we have the estimate

$$\begin{aligned}&\|\mathbf{w}_2\|_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|\pi_2\|_{H^{\frac{1}{2}+\alpha}}\\ &\leq C\Big(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} + \|h\|_{H^{\frac{1}{2}+\alpha}} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}} + \|(1-\Psi)\mathbf{F}\|_{\mathbf{L}^2} + \|(1-\Psi)h\|_{H^1}\Big).\end{aligned} \tag{2.3.8}$$

Finally, combining (2.3.5), (2.3.7) and (2.3.8) we obtain (2.3.3). This completes the proof. □

2.4 Rewriting the Stokes system as an operator equation

The goal of this section is to rewrite the system (2.1.1) as an operator equation. First, in Subsection 2.4.1, we begin by presenting some results concerning the Leray projector, which involve a careful analysis of a specific elliptic equation. In Subsection 2.4.2, we introduce the Stokes operator and prove the analyticity of its underlying semigroup. In Subsection 2.4.3, an expression of the pressure π is determined in terms of $\mathbf{w}$, $\mathbf{F}$ and $\mathbf{g}$ when $h = 0$. In Subsection 2.4.4, the operator equation characterizing (2.1.1) is introduced.

2.4.1 The Leray projector

Let us recall the following result established in [NR15, Lemma 2.2].

Proposition 2.4.1. *We have the following orthogonal decomposition*

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \overset{\perp}{\oplus} \nabla H^1_{\Gamma_n}(\Omega), \tag{2.4.1}$$

where $H^1_{\Gamma_n}(\Omega) = \{p \in H^1(\Omega) \mid p = 0 \text{ on } \Gamma_n\}$. *Moreover, the orthogonal projection* $P \in \mathcal{L}(\mathbf{L}^2(\Omega))$ *from* $\mathbf{L}^2(\Omega)$ *onto* $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$ *is characterized by*

$$P\mathbf{F} = \mathbf{F} - \nabla q_1 - \nabla q_2, \tag{2.4.2}$$

where q_1 *and* q_2 *are solutions of the following elliptic problems:*

$$\begin{aligned}&q_1 \in H^1_0(\Omega), \quad \Delta q_1 = \operatorname{div}\mathbf{F} \ \text{ in } \Omega,\\ &q_2 \in H^1_{\Gamma_n}(\Omega), \quad \Delta q_2 = 0 \ \text{ in } \Omega, \quad \frac{\partial q_2}{\partial \mathbf{n}} = (\mathbf{F} - \nabla q_1)\cdot\mathbf{n} \ \text{ on } \Gamma_d.\end{aligned} \tag{2.4.3}$$

The variational problem satisfied by $q = q_1 + q_2$ is

$$\begin{cases}\text{Find } q \in H^1_{\Gamma_n}(\Omega) \text{ such that}\\ \displaystyle\int_\Omega \nabla q\cdot\nabla\phi\,dx = \int_\Omega \mathbf{F}\cdot\nabla\phi\,dx, \quad \forall\phi \in H^1_{\Gamma_n}(\Omega).\end{cases} \tag{2.4.4}$$

For all $\mathbf{F} \in \mathbf{L}^2(\Omega)$, Problem (2.4.4) admits a unique solution and

$$\|q\|_{H^1_{\Gamma_n}} \leq C\,\|\mathbf{F}\|_{\mathbf{L}^2}. \tag{2.4.5}$$

We would like to obtain other stability estimates when $\mathbf{F} \in \mathbf{H}^s(\Omega)$ for $-\frac{1}{2} < s < \frac{1}{2} + \alpha^*$.

Because of the presence of mixed boundary conditions involved in Problem (2.4.4), it is convenient to associate to (2.4.4) a variational problem in the larger domain $\widetilde{\Omega}$ involving only Neumann boundary conditions. For a given $q \in L^2(\Omega)$, we denote by $\widetilde{q} = \mathcal{E}_o q$ the extension of q to $\widetilde{\Omega}$ defined by

$$\widetilde{q}(x) = (\mathcal{E}_o q)(x) = \begin{cases} q(x) & \text{if } x \in \Omega, \\ -q(\mathcal{S}_n x) & \text{if } x \in \mathcal{S}_n \Omega. \end{cases} \tag{2.4.6}$$

For a given $\mathbf{F} \in \mathbf{L}^2(\Omega)$, with $\mathbf{F} = F_1 \vec{\mathbf{e}}_1 + F_2 \vec{\mathbf{e}}_2 = F_n \vec{\boldsymbol{n}} + F_\tau \vec{\boldsymbol{\tau}}$ where $\vec{\boldsymbol{n}}$ is the unit normal to Γ_n outward Ω and $\vec{\boldsymbol{\tau}}$ is such that $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{\tau}})$ is an orthonormal basis of $\mathbb{R}^2$, we denote by $\widetilde{\mathbf{F}} = \mathcal{E}\mathbf{F}$ the extension of $\mathbf{F}$ to $\widetilde{\Omega}$ defined by

$$\widetilde{\mathbf{F}}(x) = (\mathcal{E}\mathbf{F})(x) = \begin{cases} \mathbf{F}(x) & \text{if } x \in \Omega, \\ F_n(\mathcal{S}_n x)\vec{\boldsymbol{n}} - F_\tau(\mathcal{S}_n x)\vec{\boldsymbol{\tau}} & \text{if } x \in \mathcal{S}_n \Omega. \end{cases} \tag{2.4.7}$$

The following proposition is a consequence of definitions (2.4.6) and (2.4.7).

Proposition 2.4.2. *Let $\mathbf{F}$ belong to $\mathbf{L}^2(\Omega)$ and let $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$ be the extension of $\mathbf{F}$ defined by (2.4.7).*

A function $q \in H^1_{\Gamma_n}(\Omega)$ is solution to (2.4.4) if and only if $\widetilde{q}$ defined in (2.4.6) is solution to

$$\begin{cases} \text{Find } \widetilde{q} \in \mathcal{H}^1(\widetilde{\Omega}) \text{ such that} \\ \displaystyle\int_{\widetilde{\Omega}} \nabla \widetilde{q} \cdot \nabla \Psi \, dx = \int_{\widetilde{\Omega}} \widetilde{\mathbf{F}} \cdot \nabla \Psi \, dx, \quad \forall \Psi \in \mathcal{H}^1(\widetilde{\Omega}). \end{cases} \tag{2.4.8}$$

Proposition 2.4.3. *Let $\mathbf{F}$ belong to $\mathbf{L}^2(\Omega)$ and let $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$ be the extension of $\mathbf{F}$ defined by (2.4.7).*

Problem (2.4.8) admits a unique solution $\widetilde{q} \in \mathcal{H}^1(\widetilde{\Omega})$ which satisfies the following estimates:

$$\|\widetilde{q}\|_{\mathcal{H}^1(\widetilde{\Omega})} \le C \, \|\widetilde{\mathbf{F}}\|_{\mathbf{L}^2(\widetilde{\Omega})}, \tag{2.4.9}$$

$$\|\widetilde{q}\|_{\mathcal{H}^{\frac{3}{2}+\alpha}(\widetilde{\Omega})} \le C_\alpha \, \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\widetilde{\Omega})}, \quad \forall \alpha \in (0, \alpha^*), \tag{2.4.10}$$

$$\|\widetilde{q}\|_{\mathcal{H}^{s+1}(\widetilde{\Omega})} \le C_s \, \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^s(\widetilde{\Omega})}, \quad \forall s \in (-1/2, 1/2 + \alpha^*). \tag{2.4.11}$$

Before proving that proposition, to show (2.4.11) when $s \in (-1/2, 0)$, we are going to use the transposition method. For that, we introduce the adjoint problem

$$\Delta \chi = \zeta \text{ in } \widetilde{\Omega}, \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \text{ on } \widetilde{\Gamma} = \partial \widetilde{\Omega}, \tag{2.4.12}$$

with $\zeta \in L^2_0(\widetilde{\Omega})$.

Lemma 2.4.1. *The variational solution to (2.4.12) satisfies*

$$\begin{aligned} &\|\chi\|_{\mathcal{H}^{\frac{3}{2}+\alpha}(\widetilde{\Omega})} \le C_\alpha \, \|\zeta\|_{\mathcal{H}^{-\frac{1}{2}+\alpha}(\widetilde{\Omega})}, \quad \forall \alpha \in (0, \alpha^*), \\ &\|\chi\|_{\mathcal{H}^1(\widetilde{\Omega})} \le C_\alpha \, \|\zeta\|_{(\mathcal{H}^1(\widetilde{\Omega}))'}, \\ &\textit{and} \\ &\|\chi\|_{\mathcal{H}^{1+s}(\widetilde{\Omega})} \le C_s \, \|\zeta\|_{\mathcal{H}^{s-1}(\widetilde{\Omega})}, \quad \forall s \in [0, 1/2 + \alpha^*). \end{aligned} \tag{2.4.13}$$

Proof. Estimate $(2.4.13)_1$ is a consequence of [Dau88, Chapter 8, Corollary 23.5, page 197]. Estimate $(2.4.13)_2$ is a consequence of the Lax-Milgram Lemma. Estimate $(2.4.13)_3$ can be obtained by interpolation from $(2.4.13)_1$ and $(2.4.13)_2$. □

When $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$, it is possible to check that $\widetilde{q}$ is solution to (2.4.8) if and only if, for all $s \in (-1/2, 0]$, $\widetilde{q}$ is solution to

$$\begin{aligned}
&\text{Find } \widetilde{q} \in \mathcal{H}^{s+1}(\widetilde{\Omega}) \text{ such that} \\
&\langle \widetilde{q}, \zeta \rangle_{\mathcal{H}^{s+1}(\widetilde{\Omega}), \mathcal{H}^{-s-1}(\widetilde{\Omega})} = -\langle \widetilde{\mathbf{F}}, \nabla \chi \rangle_{\mathbf{H}^s(\widetilde{\Omega}), \mathbf{H}^{-s}(\widetilde{\Omega})}, \quad \forall \zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}),
\end{aligned} \tag{2.4.14}$$

where χ is the solution to (2.4.12).

Lemma 2.4.2. *For all $s \in (-1/2, 0]$, Problem (2.4.14) admits a unique solution and*

$$\|\widetilde{q}\|_{\mathcal{H}^{s+1}(\widetilde{\Omega})} \leq C_s \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^s(\widetilde{\Omega})}. \tag{2.4.15}$$

Proof. When $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$, for any $s \in (-1/2, 0]$ (2.4.22) follows from (2.4.14) and (2.4.13). Indeed, since $-s-1 \in (-1, -1/2)$, Lemma 2.4.1 implies that the solution χ of system (2.4.12) belongs to $\mathcal{H}^{-s+1}(\widetilde{\Omega})$. Furthermore, $\|\chi\|_{\mathcal{H}^{-s+1}(\widetilde{\Omega})} \leq C_s \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})}$. Then, this estimate together with (2.4.14) allows us to obtain

$$\begin{aligned}
\|\widetilde{q}\|_{\mathcal{H}^{s+1}(\widetilde{\Omega})} &= \sup_{\substack{\zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}) \\ \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} = 1}} \left| \langle \widetilde{q}, \zeta \rangle_{\mathcal{H}^{s+1}(\widetilde{\Omega}), \mathcal{H}^{-s-1}(\widetilde{\Omega})} \right| \\
&= \sup_{\substack{\zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}) \\ \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} = 1}} \left| \langle \widetilde{\mathbf{F}}, \nabla \chi \rangle_{\mathbf{H}^s(\widetilde{\Omega}), \mathbf{H}^{-s}(\widetilde{\Omega})} \right| \\
&\leq \sup_{\substack{\zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}) \\ \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} = 1}} \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^s(\widetilde{\Omega})} \|\nabla \chi\|_{\mathbf{H}^{-s}(\widetilde{\Omega})} \\
&\leq \sup_{\substack{\zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}) \\ \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} = 1}} \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^s(\widetilde{\Omega})} \|\chi\|_{\mathcal{H}^{-s+1}(\widetilde{\Omega})} \\
&\leq \sup_{\substack{\zeta \in \mathcal{H}^{-s-1}(\widetilde{\Omega}) \\ \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} = 1}} \|\widetilde{\mathbf{F}}\|_{\mathbf{H}^s(\widetilde{\Omega})} \|\zeta\|_{\mathcal{H}^{-s-1}(\widetilde{\Omega})} \\
&\leq C_s \|\mathbf{F}\|_{\mathbf{H}^s(\widetilde{\Omega})}.
\end{aligned} \tag{2.4.16}$$

This proves the estimate (2.4.22) when $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$. From (2.4.22) when $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\widetilde{\Omega})$, it follows that if Problem (2.4.14) admits a solution, this solution is necessarily unique. To prove the existence of solution to Problem (2.4.14) for a given $\widetilde{\mathbf{F}} \in \mathbf{H}^s(\widetilde{\Omega})$, with $s \in (-1/2, 0)$, we use a density argument. We approximate $\widetilde{\mathbf{F}}$ in $\mathbf{H}^s(\widetilde{\Omega})$ by a sequence $(\widetilde{\mathbf{F}}_k)_k$ in $\mathbf{L}^2(\widetilde{\Omega})$ converging to $\widetilde{\mathbf{F}}$ in $\mathbf{H}^s(\widetilde{\Omega})$. We denote by $\widetilde{q}_k$ the solution to (2.4.14) or to (2.4.8) corresponding to $\widetilde{\mathbf{F}}_k$. Since $\widetilde{q}_k - \widetilde{q}_m$ satisfies

$$\|\widetilde{q}_k - \widetilde{q}_m\|_{\mathcal{H}^{s+1}(\widetilde{\Omega})} \leq C_s \|\widetilde{\mathbf{F}}_k - \widetilde{\mathbf{F}}_m\|_{\mathbf{H}^s(\widetilde{\Omega})},$$

the sequence $(\widetilde{q}_k)_k$ is a Cauchy sequence in $\mathcal{H}^{s+1}(\widetilde{\Omega})$. Its limit $\widetilde{q}$ in $\mathcal{H}^{s+1}(\widetilde{\Omega})$ is solution to (2.4.14) and it satisfies (2.4.22). $\square$

We are now in position to prove Proposition 2.4.3.

Proof of Proposition 2.4.3. The existence and uniqueness of $\widetilde{q} \in \mathcal{H}^1(\widetilde{\Omega})$ solution to (2.4.8) together with estimate (2.4.9) follow from the Lax-Milgram Lemma.

Estimate (2.4.10) follows from [Dau88, Chapter 8, Corollary 23.5, page 197].

For $s \in [0, 1/2 + \alpha^*)$, Estimate (2.4.11) can be obtained by interpolation between (2.4.9) and

(2.4.10).

Estimate (2.4.11) for $s \in (-1/2, 0)$ follows from Lemma 2.4.2. This completes the proof of Proposition 2.4.3.

The following proposition is a direct consequence of Propositions 2.4.2 and 2.4.3.

Proposition 2.4.4. *Let* $\mathbf{F}$ *belong to* $\mathbf{L}^2(\Omega)$. *In addition to* (2.4.5), *the solution* $q \in H^1_{\Gamma_n}(\Omega)$ *to Problem* (2.4.4) *satisfies*

$$\|q\|_{H^{\frac{3}{2}+\alpha}(\Omega)} \leq C_\alpha \|\mathbf{F}\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)}, \quad \forall \alpha \in (0, \alpha^*), \tag{2.4.17}$$

$$\|q\|_{H^{s+1}(\Omega)} \leq C_s \|\mathbf{F}\|_{\mathbf{H}^s(\Omega)}, \quad \forall s \in (-1/2, 1/2 + \alpha^*). \tag{2.4.18}$$

We can also use the transposition method to extend the notion of solution to the variational problem (2.4.4) when $\mathbf{F} \in \mathbf{H}^s(\Omega)$ with $s \in (-1/2, 0)$. For that, we introduce the adjoint problem

$$\Delta\chi = \zeta \text{ in } \Omega, \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d, \quad \chi = 0 \text{ on } \Gamma_n, \tag{2.4.19}$$

with $\zeta \in L^2(\Omega)$.

Lemma 2.4.3. *The variational solution to* (2.4.19) *satisfies*

$$\begin{aligned}
&\|\chi\|_{H^{\frac{3}{2}+\alpha}(\Omega)} \leq C_\alpha \|\zeta\|_{H^{-\frac{1}{2}+\alpha}(\Omega)}, \quad \forall \alpha \in (0, \alpha^*), \\
&\|\chi\|_{H^1(\Omega)} \leq C_\alpha \|\zeta\|_{(H^1_{\Gamma_n}(\Omega))'}, \\
&\textit{and} \\
&\|\chi\|_{H^{1+s}(\Omega)} \leq C_s \|\zeta\|_{H^{s-1}(\Omega)}, \quad \forall s \in [0, 1/2 + \alpha^*).
\end{aligned} \tag{2.4.20}$$

Proof. The result follows from Lemma 2.4.1 and a symmetry argument as in Proposition 2.4.2. □

When $\mathbf{F} \in \mathbf{L}^2(\Omega)$, it is possible to check that q is solution to (2.4.4) if and only if q is solution to

$$\begin{aligned}
&\text{Find } q \in H^{s+1}(\Omega) \text{ such that} \\
&\langle q, \zeta \rangle_{H^{s+1}, H^{-s-1}} = -\langle \mathbf{F}, \nabla\chi \rangle_{\mathbf{H}^s, \mathbf{H}^{-s}}, \quad \forall \zeta \in H^{-s-1}(\Omega),
\end{aligned} \tag{2.4.21}$$

where χ is the solution to (2.4.19), and $s \in (-1/2, 0]$.

Lemma 2.4.4. *For all* $s \in (-1/2, 0]$, *Problem* (2.4.21) *admits a unique solution and*

$$\|q\|_{H^{s+1}(\Omega)} \leq C_s \|\mathbf{F}\|_{\mathbf{H}^s(\Omega)}, \quad \forall s \in (-1/2, 0]. \tag{2.4.22}$$

Proof. The proof is similar to that of Lemma 2.4.2. □

The following corollary is a direct consequence of Proposition 2.4.4.

Corollary 2.4.1. *The operator* $P \in \mathcal{L}(\mathbf{L}^2(\Omega))$ *defined by*

$$P\mathbf{F} = \mathbf{F} - \nabla q, \tag{2.4.23}$$

where q *is the solution to* (2.4.4) *or to* (2.4.21), *is also continuous from* $\mathbf{H}^s(\Omega)$ *into itself for all* $s \in [0, 1/2 + \alpha^*)$. *It can also be extended by continuity to a bounded operator in* $\mathbf{H}^s(\Omega)$ *for all* $s \in (-1/2, 0)$.

2.4.2 Stokes operator and the analyticity of its underlying semigroup

For $0 < s < 1/2$, we introduce the space

$$\mathbf{V}^s_{n,\Gamma_d}(\Omega) = \mathbf{H}^s_{\Gamma_d}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega), \tag{2.4.24}$$

equipped with the $\mathbf{H}^s_{\Gamma_d}$ Sobolev norm, and we define $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$ as the dual of $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$ with $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ as pivot space, equipped with the dual norm of $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$.

Lemma 2.4.5. *Let $s \in (0, 1/2)$. The following assertions hold:*

(i) $P\left(\mathbf{H}^s_{\Gamma_d}(\Omega)\right) = \mathbf{V}^s_{n,\Gamma_d}(\Omega)$.

(ii) There exists an isomorphism between the spaces $\overline{\mathbf{V}^0_{n,\Gamma_d}(\Omega)}^{\|\cdot\|_{\mathbf{H}^{-s}_{\Gamma_d}(\Omega)}}$ *and* $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$. *Thus, the space* $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$ *can be identified with a closed subspace of* $\mathbf{H}^{-s}_{\Gamma_d}(\Omega)$.

Proof.

(i) The inclusion $\supset$ follows directly from the definition of the space $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$ (see (2.4.24)). The other inclusion is a consequence of Corollary 2.4.1.

(ii) It suffices to prove that the norms $\|\cdot\|_{\mathbf{H}^{-s}_{\Gamma_d}}$ and $\|\cdot\|_{\mathbf{V}^{-s}_{n,\Gamma_d}}$ are equivalent in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Indeed, if the two norms are equivalents in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$, the result follows from the fact that the space $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is dense in $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$. Let us now show the equivalence of the norms in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$. We follow the approach used in [GS11, p. 246]. We split the proof into two steps.

<u>*Step 1*</u>: Let us assume that $\mathbf{v} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Then, using assertion (i), we obtain

$$\|\mathbf{v}\|_{\mathbf{V}^{-s}_{n,\Gamma_d}} = \sup_{\mathbf{w}\in \mathbf{V}^s_{n,\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{L}^2}|}{\|\mathbf{w}\|_{\mathbf{H}^s_{\Gamma_d}}} \le \sup_{\mathbf{w}\in \mathbf{H}^s_{\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{L}^2}|}{\|\mathbf{w}\|_{\mathbf{H}^s_{\Gamma_d}}} = \|\mathbf{v}\|_{\mathbf{H}^{-s}_{\Gamma_d}}.$$

<u>*Step 2*</u>: Let $\alpha \in (0, \alpha^*)$. Let us assume that $\mathbf{v} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Then, using continuity of the Leray projector P from $\mathbf{H}^s_{\Gamma_d}(\Omega)$ into $\mathbf{H}^s_{\Gamma_d}(\Omega)$ (see Corollary 2.4.1), we obtain

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{H}^{-s}_{\Gamma_d}} = \sup_{\mathbf{w}\in \mathbf{H}^s_{\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{L}^2}|}{\|\mathbf{w}\|_{\mathbf{H}^s_{\Gamma_d}}} &= \sup_{\mathbf{w}\in \mathbf{H}^s_{\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, P\mathbf{w}\rangle_{\mathbf{L}^2}|}{\|\mathbf{w}\|_{\mathbf{H}^s_{\Gamma_d}}} \\
&\le C \sup_{\mathbf{w}\in \mathbf{H}^s_{\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, P\mathbf{w}\rangle_{\mathbf{L}^2}|}{\|P\mathbf{w}\|_{\mathbf{H}^s_{\Gamma_d}}} \\
&\le C \sup_{\mathbf{z}\in \mathbf{V}^s_{n,\Gamma_d}(\Omega)} \frac{|\langle \mathbf{v}, z\rangle_{\mathbf{L}^2}|}{\|z\|_{\mathbf{H}^s_{\Gamma_d}}} \\
&= C\|\mathbf{v}\|_{\mathbf{V}^{-s}_{n,\Gamma_d}}.
\end{aligned}$$

This completes the proof.

□

The Stokes operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ that we will consider is defined by

$$\begin{aligned}
\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)) = \Big\{&\mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \cap \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists \pi \in H^{\frac{1}{2}+\alpha}(\Omega) \text{ such that } \operatorname{div}\sigma(\mathbf{w}, \pi) \in \mathbf{L}^2(\Omega) \\
&\text{and } \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \text{ on } \Gamma_n\Big\},
\end{aligned}$$

$$A_0\mathbf{w} = P \operatorname{div}\sigma(\mathbf{w}, \pi).$$

For all $\theta \in (\pi/2, \pi)$, let us define the sector Σ_θ by

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \theta\}.$$

Theorem 2.4.1. *There exist $\theta_0 \in (\pi/2, \pi)$ and $C > 0$ such that*

$$\|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(\mathbf{V}^0_{n,\Gamma_d}(\Omega))} \leq \frac{C}{|\lambda|}, \quad \textit{for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\}.$$

In particular, the unbounded operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$.

Proof. According to [DL99, Proposition 3, p. 380], it suffices to show that there exist $\lambda_0 \geq 0$ and $C_0 > 0$ such that

$$a_{\lambda_0}(\mathbf{v}, \mathbf{v}) = ((\lambda_0 I - A_0)\mathbf{v}, \mathbf{v})_{\mathbf{L}^2(\Omega)} \geq C_0 \|\mathbf{v}\|^2_{\mathbf{V}^1_{\Gamma_d}(\Omega)} \text{ for all } \mathbf{v} \in \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)).$$

The previous inequality is verified with $\lambda_0 = 1$ and $C_0 = C_0(\nu) > 0$. □

For all $\alpha \in (0, \alpha^*)$, we introduce the heterogeneous space

$$\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}(\Omega) := \left\{\mathbf{v} \in \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega) \mid (1-\Psi)\mathbf{v} \in \mathbf{L}^2(\Omega)\right\} \tag{2.4.25}$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}} := \left(\|\mathbf{v}\|^2_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} + \|(1-\Psi)\mathbf{v}\|^2_{\mathbf{L}^2}\right)^{\frac{1}{2}}, \tag{2.4.26}$$

for all $\mathbf{v} \in \mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}(\Omega)$. Here, Ψ is the cut-off function introduced in (2.2.6).

Proposition 2.4.5. *Let $\alpha \in (0, \alpha^*)$. The space $\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}(\Omega)$, equipped with the norm $\|\cdot\|_{\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}}$ defined in (2.4.26), is a Hilbert space.*

Proof. Let $(\mathbf{v}_k)_k$ be a Cauchy sequence in $\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}(\Omega)$. Next, since the spaces $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$ and $\mathbf{L}^2(\Omega)$ are complete, using Lemma 2.4.5, there exist $\mathbf{v} \in \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$ and $\mathbf{u} \in \mathbf{L}^2(\Omega)$ such that

$$\mathbf{v}_k \xrightarrow[k\to+\infty]{} \mathbf{v} \text{ in } \mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega) \tag{2.4.27}$$

and

$$(1-\Psi)\mathbf{v}_k \xrightarrow[k\to+\infty]{} \mathbf{u} \text{ in } \mathbf{L}^2(\Omega). \tag{2.4.28}$$

From (2.4.27) we deduce that

$$(1-\Psi)\mathbf{v}_k \xrightarrow[k\to+\infty]{} (1-\Psi)\mathbf{v} \text{ in } \mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega). \tag{2.4.29}$$

Then, from (2.4.28) and (2.4.29) and the uniqueness of the limit we deduce that $\mathbf{u} = (1-\Psi)\mathbf{v}$. This completes the proof. □

According to Corollary 2.4.1, the Leray projector defined by (2.4.23) belongs to $\mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))$ for all $\alpha \in (0, \alpha^*)$.

Proposition 2.4.6. *For all $\alpha \in (0, \alpha^*)$, the Leray projector $P \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))$ also belongs to $\mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$.*

Proof. We will show that there exists a constant $C > 0$ such that

$$\|P\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}} + \|(1-\Psi)P\mathbf{F}\|_{\mathbf{L}^2} \le C\left(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}} + \|(1-\Psi)\mathbf{F}\|_{\mathbf{L}^2}\right), \tag{2.4.30}$$

for all $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$. Since $P \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))$, in order to obtain (2.4.30) it is sufficient to prove that

$$\|(1-\Psi)P\mathbf{F}\|_{\mathbf{L}^2} \le C\left(\|(1-\Psi)\mathbf{F}\|_{\mathbf{L}^2} + \|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}}\right). \tag{2.4.31}$$

We notice that

$$(1-\Psi)P\mathbf{F} = (1-\Psi)\mathbf{F} - \nabla((1-\Psi)q) - \nabla\Psi q, \tag{2.4.32}$$

where q is solution to (2.4.21) with $s = -\frac{1}{2} + \alpha$. Next, from Lemma 2.4.6 below, we obtain that $q \in H^{\frac{1}{2}+\alpha,1}(\Omega)$ and then, thanks to (2.4.32) we deduce the estimate (2.4.31). □

Lemma 2.4.6. *If $\mathbf{F}$ belongs to $\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$ with $\alpha \in (0, \alpha^*)$, then the solution q to (2.4.21) with $s = -\frac{1}{2} + \alpha$ belongs to $H^{\frac{1}{2}+\alpha,1}(\Omega)$ and satisfies*

$$\|q\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} \le C_\alpha \|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)}, \quad \forall \alpha \in (0, \alpha^*). \tag{2.4.33}$$

Proof. Firstly, since $\|q\|_{H^{\frac{1}{2}+\alpha}(\Omega)} \le C_\alpha \|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)}$ for all $\alpha \in (0, 1/2)$ (see Proposition 2.4.4), it suffices to show that

$$\|(1-\Psi)q\|_{H^1} \le C_\alpha \|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}}, \quad \forall \alpha \in (0, \alpha^*). \tag{2.4.34}$$

The function $p = (1-\Psi)q$ is solution to the equation

$$\begin{aligned}
&\Delta p = \operatorname{div}((1-\Psi)\mathbf{F}) + q\Delta(1-\Psi) - 2\nabla q \cdot \nabla\Psi + \mathbf{F}\cdot\nabla\Psi \quad \text{in } \Omega,\\
&\frac{\partial p}{\partial \mathbf{n}} = (1-\Psi)\mathbf{F}\cdot\mathbf{n} - q\frac{\partial \Psi}{\partial \mathbf{n}} \quad \text{on } \Gamma_d, \quad p = 0 \quad \text{on } \Gamma_n.
\end{aligned}$$

We write $p = p_1 + p_2$, where p_1 is the solution to

$$\Delta p_1 = \operatorname{div}((1-\Psi)\mathbf{F}) \quad \text{in } \Omega, \quad \frac{\partial p_1}{\partial \mathbf{n}} = (1-\Psi)\mathbf{F}\cdot\mathbf{n} \quad \text{on } \Gamma_d, \quad p_1 = 0 \quad \text{on } \Gamma_n,$$

and by p_2 the solution to

$$\Delta p_2 = -q\Delta\Psi - 2\nabla q\cdot\nabla\Psi - \mathbf{F}\cdot\nabla\Psi \quad \text{in } \Omega, \quad \frac{\partial p_2}{\partial \mathbf{n}} = -q\frac{\partial \Psi}{\partial \mathbf{n}} \quad \text{on } \Gamma_d, \quad p_2 = 0 \quad \text{on } \Gamma_n.$$

Since

$$\|(1-\Psi)\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)},$$

the needed estimate for p_1 in $H^1(\Omega)$ follows from (2.4.5). Next from

$$\|q\Delta(1-\Psi) - 2\nabla q\cdot\nabla\Psi - \mathbf{F}\cdot\nabla\Psi\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)} \le C\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)},$$

Lemma 2.4.3 and Lax-Milgram Lemma we deduce that $\|p_2\|_{H^1(\Omega)} \le C\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)}$. The proof is complete. □

In Lemma 2.4.7 below, we show that $[\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^{-1}_{\Gamma_d}(\Omega)]_s = \mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$ for all $s \in (0, 1/2)$. The proof of this lemma relies on an interpolation result in the Dirichlet case proved in [MM08, Theorem 2.12]. More precisely, by using an extension and symmetry argument, we reduce the problem to the Dirichlet case.

Let $\widetilde{\Omega}$ defined by

$$\widetilde{\Omega} = \Omega \cup \Gamma_n \cup \widehat{\Omega}. \tag{2.4.35}$$

The construction of $\widehat{\Omega}$ (see Figure 2.3) is given below.

1. We first construct the symmetric domain $S_n\Omega$ of Ω with respect to the line D_n containing the segment Γ_n.
2. From assumption $(H4)$ we know that there exist in particular two half-lines ℓ_1 and ℓ_2 passing through the endpoints A and E of the segment Γ_n, such that $\ell_1 \cap \overline{\Omega} = \{A\}$ and $\ell_2 \cap \overline{\Omega} = \{E\}$. The choice of these half-lines is not unique. We denote by ℓ_1' and ℓ_2' the symmetric half-lines of ℓ_1 and ℓ_2, respectively, with respect to D_n.
3. We now select an arbitrary point $B \neq A$ on the half-line ℓ_1', and then define the point C such that $AC \perp BC$. We denote by D the midpoint of the segment BC. Similarly, we select a point $F \neq E$ on the half-line ℓ_2' and define G such that $EG \perp FG$. We then denote by H the midpoint of the segment FG.
4. We define the segments AI and EJ, passing through D and H, respectively, in such a way that the segment IJ does not intersect the domain Ω.
5. Finally, we define $\widehat{\Omega}$ as the polygon $\widehat{\Omega} = AIJE$ (see Figure 2.3).

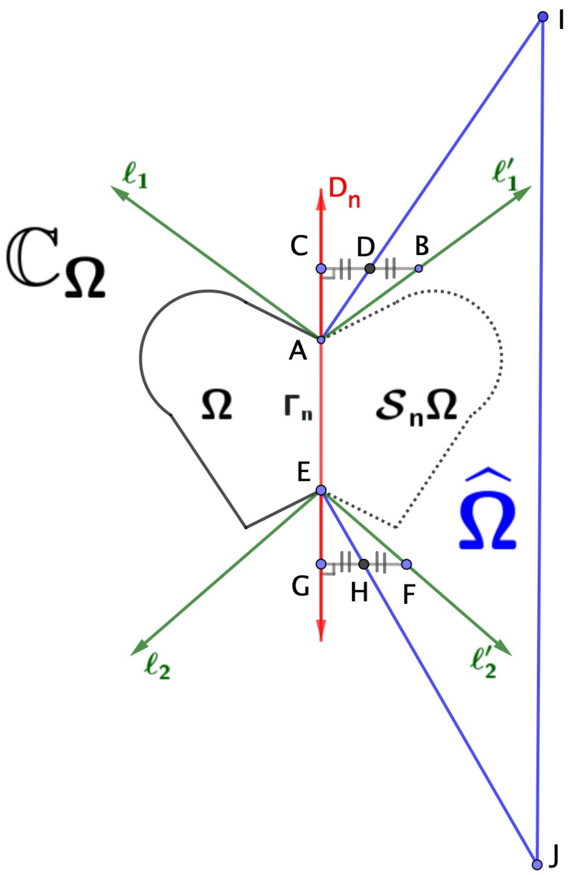


Figure 2.3 – Construction of $\widehat{\Omega}$.

Remark 4. We highlight that the particular construction of $\widehat{\Omega}$ is motivated by the definition of the extension operator $\mathcal{E}^s$ (see (2.4.37)). Specifically, thanks to the step 4, where the segments

AI and EJ are constructed in a such way that they pass for the midpoints of the segments BC and FG, respectively, allows us to preserve a desired trace property. See proof of Lemma 2.4.7 below.

We first introduce the following spaces:

$$\mathbf{V}_n^0(\widetilde{\Omega}) = \left\{ \widetilde{\mathbf{v}} \in \mathbf{L}^2(\widetilde{\Omega}) \mid \operatorname{div} \widetilde{\mathbf{v}} = 0 \text{ in } \widetilde{\Omega},\ \widetilde{\mathbf{v}} \cdot \mathbf{n} = 0 \text{ on } \partial\widetilde{\Omega} \right\},$$
$$\mathbf{V}_0^1(\widetilde{\Omega}) = \mathbf{H}_0^1(\widetilde{\Omega}) \cap \mathbf{V}_n^0(\widetilde{\Omega}) \text{ and } \mathbf{V}_n^s(\widetilde{\Omega}) = \mathbf{H}^s(\widetilde{\Omega}) \cap \mathbf{V}_n^0(\widetilde{\Omega}).$$

Lemma 2.4.7. *Let $s \in (0, 1/2)$. The following assertions hold.*

(i) *There exists an extension operator E satisfying*

$$\begin{aligned} &E \in \mathcal{L}(\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_n^0(\widetilde{\Omega})) \cap \mathcal{L}(\mathbf{V}_{\Gamma_d}^1(\Omega), \mathbf{V}_0^1(\widetilde{\Omega})),\\ &E \in \mathcal{L}(\mathbf{V}_{n,\Gamma_d}^s(\Omega), \mathbf{V}_n^s(\widetilde{\Omega})), \end{aligned} \tag{2.4.36}$$

where $\widetilde{\Omega}$ is defined in (2.4.35).

(ii) $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^1(\Omega)]_s = \mathbf{V}_{n,\Gamma_d}^s(\Omega)$ *and* $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_s = \mathbf{V}_{n,\Gamma_d}^{-s}(\Omega)$.

Proof.

(i) After a possible rotation of the domain Ω, we assume that the origin of the coordinate system is located at Γ_n. We denote by $\mathbb{C}_\Omega$ the half-plane containing Ω (this is possible thanks to assumption $(H4)$ stated in Subsection 2.2.1). We also denote by $\mathcal{E}^z : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\mathbb{C}_\Omega)$ the zero extension operator. We now introduce the operator $\mathcal{E}^s : \mathbf{L}^2(\mathbb{C}_\Omega) \longrightarrow \mathbf{L}^2(\widetilde{\Omega})$ defined by

$$(\mathcal{E}^s\mathbf{u})(z_1, z_2) = \begin{cases} \mathbf{u}(z_1, z_2) & \text{if } (z_1, z_2) \in \mathbb{C}_\Omega,\\ \widetilde{\mathbf{u}}(z_1, z_2) & \text{if } (z_1, z_2) \in \widetilde{\Omega} \setminus \Omega, \end{cases} \tag{2.4.37}$$

where $\widetilde{\mathbf{u}}(z_1, z_2) = (\widetilde{u}_1, \widetilde{u}_2)$ with

$$\begin{aligned} \widetilde{u}_1(z_1, z_2) &= 3u_1(-z_1, z_2) - 2u_1(-2z_1, z_2),\\ \widetilde{u}_2(z_1, z_2) &= -3u_2(-z_1, z_2) + 4u_2(-2z_1, z_2). \end{aligned} \tag{2.4.38}$$

We then define the operator $E : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\widetilde{\Omega})$ by $E = \mathcal{E}^s \circ \mathcal{E}^z$. Let us notice that the following equality holds in the sense of distributions:

$$\operatorname{div} \widetilde{\mathbf{u}}(z_1, z_2) = -3\,(\operatorname{div} \mathbf{u})\,(-z_1, z_2) + 4\,(\operatorname{div} \mathbf{u})\,(-2z_1, z_2).$$

Thus, in particular, if $\operatorname{div} \mathbf{u} = 0$, then $\operatorname{div} \widetilde{\mathbf{u}} = 0$. On the other hand, if $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\mathbf{u}|_{\Gamma_d} = 0$, then $\widetilde{\mathbf{u}}|_{\partial\widehat{\Omega}\setminus\Gamma_n} = 0$. We remark that, at this point, we use the particular construction of $\widehat{\Omega}$.

Thanks to this construction of the operator E, we can verify (2.4.36).

(ii) Let us first observe that $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_s = \mathbf{V}_{n,\Gamma_d}^{-s}(\Omega)$ is a consequence of the equality $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^1(\Omega)]_s = \mathbf{V}_{n,\Gamma_d}^s(\Omega)$ and [Tar07, Lemma 41.3, p. 196], since

$$\mathbf{V}_{n,\Gamma_d}^{-s}(\Omega) = (\mathbf{V}_{n,\Gamma_d}^s(\Omega))' = [\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_s \text{ for all } s \in (0, 1/2).$$

Let us show that $[\mathbf{V}_{n,\Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^1(\Omega)]_s = \mathbf{V}_{n,\Gamma_d}^s(\Omega)$ for all $s \in (0, 1/2)$.

Let us first consider the restriction operator R satisfying

$$R \in \mathcal{L}(\mathbf{V}_n^0(\widetilde{\Omega}), \mathbf{V}_{n,\Gamma_d}^0(\Omega)) \cap \mathcal{L}(\mathbf{V}_0^1(\widetilde{\Omega}), \mathbf{V}_{\Gamma_d}^1(\Omega)). \tag{2.4.39}$$

- $[\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^1_{\Gamma_d}(\Omega)]_s \subset \mathbf{V}^s_{n,\Gamma_d}(\Omega)$.

  Let $\mathbf{v} \in [\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^1_{\Gamma_d}(\Omega)]_s$. Thanks to (2.4.36) we obtain that $E\mathbf{v} \in [\mathbf{V}^0_n(\widetilde{\Omega}), \mathbf{V}^1_0(\widetilde{\Omega})]_s$. But, from [MM08, Theorem 2.12] we know that $[\mathbf{V}^0_n(\widetilde{\Omega}), \mathbf{V}^1_0(\widetilde{\Omega})]_s = \mathbf{V}^s_n(\widetilde{\Omega})$. Thus, $E\mathbf{v} \in \mathbf{V}^s_n(\widetilde{\Omega})$ and then in particular $\mathbf{v} \in \mathbf{V}^s_{n,\Gamma_d}(\Omega)$.

- $\mathbf{V}^s_{n,\Gamma_d}(\Omega) \subset [\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^1_{\Gamma_d}(\Omega)]_s$.

  Let $\mathbf{v} \in \mathbf{V}^s_{n,\Gamma_d}(\Omega)$. Then from (2.4.36) it follows that $E\mathbf{v} \in \mathbf{V}^s_n(\widetilde{\Omega})$. Once again, by invoking [MM08, Theorem 2.12], we deduce that $E\mathbf{v} \in [\mathbf{V}^0_n(\widetilde{\Omega}), \mathbf{V}^1_0(\widetilde{\Omega})]_s$. Then, $\mathbf{v} = R(E\mathbf{v}) \in [\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^1_{\Gamma_d}(\Omega)]_s$.

$\square$

Let us now introduce the Stokes operator on $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.

Using Lemma 2.4.7 and the fact that A_0 is an isomorphism from $\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega))$ into $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and from $\mathcal{D}(A_0; \mathbf{V}^{-1}_{\Gamma_d}(\Omega))$ into $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ (this follows from the Lax-Milgram theorem), we deduce that A_0 is also an isomorphism from

$$\begin{aligned} &\mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)) := [\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathcal{D}(A_0; \mathbf{V}^{-1}_{\Gamma_d}(\Omega))]_{\frac{1}{2}-\alpha} \\ &\text{into } [\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^{-1}_{\Gamma_d}(\Omega)]_{\frac{1}{2}-\alpha} = \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega). \end{aligned} \tag{2.4.40}$$

But, since $\mathcal{D}(A_0; \mathbf{V}^{-1}_{\Gamma_d}(\Omega)) = \mathbf{V}^1_{\Gamma_d}(\Omega)$,

$$\mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)) = [\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathbf{V}^1_{\Gamma_d}(\Omega)]_{\frac{1}{2}-\alpha}.$$

We now establish the sectoriality of the Stokes operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ in $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.

Proposition 2.4.7. *Let $\alpha \in (0, \alpha^*)$. There exist $\theta_0 \in (\pi/2, \pi)$ and $C > 0$ such that*

$$\|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega))} \le \frac{C}{|\lambda|}, \quad \textit{for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\}. \tag{2.4.41}$$

Proof. We split the proof into three steps.

<u>*Step 1:*</u> (*Estimate in* $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$−norm). Since $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ (see Theorem 2.4.1), there exist $C_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ such that

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}^0_{n,\Gamma_d}} \le \frac{C_0}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}^0_{n,\Gamma_d}}, \tag{2.4.42}$$

for all $\lambda \in \Sigma \setminus \{0\}$ and for all $\mathbf{F} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$.

<u>*Step 2:*</u> (*Estimate in* $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$−norm). We claim that there exists $C_1 > 0$ such that

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}} \le \frac{C_1}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}} \tag{2.4.43}$$

for all $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$ and for all $\mathbf{F} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Indeed, let $\mathbf{u} := (\lambda I - A_0)^{-1}\mathbf{F}$. Then,

$$\begin{cases} \lambda \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, q) = \mathbf{F} \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \Gamma_d, \quad \sigma(\mathbf{u}, q)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{2.4.44}$$

Thus, $\mathbf{u}$ satisfies in particular

$$\lambda \int_\Omega \mathbf{u} \cdot \overline{\varphi} = -2\nu \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\overline{\varphi}) + \langle \mathbf{F}, \overline{\varphi} \rangle_{\mathbf{V}^{-1}_{\Gamma_d}, \mathbf{V}^1_{\Gamma_d}} \quad \text{for all } \overline{\varphi} \in \mathbf{V}^1_{\Gamma_d}(\Omega). \tag{2.4.45}$$

Using the same idea employed to prove coercivity in Therorem 2.4.4, it is possible to show that there exists a constant $C > 0$, independent of λ, such that

$$\|\mathbf{u}\|_{\mathbf{V}^1_{\Gamma_d}} \leq C \|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}}. \tag{2.4.46}$$

We now introduce the set

$$\mathcal{D} := \left\{ \varphi \in \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \|\varphi\|_{\mathbf{V}^1_{\Gamma_d}} \leq 1 \right\}.$$

By applying Cauchy-Schwarz inequality on the first term on the right-hand side of (2.4.45), and then using estimate (2.4.46), we obtain

$$\begin{aligned} |\lambda| \|\mathbf{u}\|_{\mathbf{V}^{-1}_{\Gamma_d}} &= |\lambda| \sup_{\varphi \in \mathcal{D}} \left| \langle \mathbf{u}, \overline{\varphi} \rangle_{\mathbf{V}^{-1}_{\Gamma_d}, \mathbf{V}^1_{\Gamma_d}} \right| \\ &\leq 2\nu \sup_{\varphi \in \mathcal{D}} \int_\Omega |\varepsilon(\mathbf{u})| |\varepsilon(\overline{\varphi})| + \sup_{\varphi \in \mathcal{D}} \left| \langle \mathbf{F}, \overline{\varphi} \rangle_{\mathbf{V}^{-1}_{\Gamma_d}, \mathbf{V}^1_{\Gamma_d}} \right| \\ &\leq C \|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}}, \end{aligned} \tag{2.4.47}$$

from where we deduce the estimate (2.4.43).

<u>*Step 3.*</u> *(Conclusion).* From Lemma 2.4.7 and the interpolation between (2.4.42) and (2.4.43) we deduce that for any $\epsilon \in (0, 1/2)$

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}^{-\epsilon}_{\Gamma_{n,\Gamma_d}}} \leq C_0 \left(\frac{C_1}{C_0} \right)^\epsilon \frac{1}{|\lambda|} \|\mathbf{F}\|_{\mathbf{V}^{-\epsilon}_{\Gamma_{n,\Gamma_d}}} \quad \text{for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\} \text{ and for all } \mathbf{F} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega). \tag{2.4.48}$$

Now, taking $\epsilon = 1/2 - \alpha \in (0, 1/2)$ in (2.4.48) and using the fact that $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is dense in $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$ we deduce that

$$\|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(\mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_{n,\Gamma_d}})} \leq \frac{C}{|\lambda|} \text{ for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\}. \tag{2.4.49}$$

□

Proposition 2.4.8. *The domain $\mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_{n,\Gamma_d}}(\Omega))$ of the Stokes operator A_0 is dense in $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.*

Proof. Since $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$ is a Hilbert space, according to Proposition 2.4.9 presented in the Appendix B at the end of this chapter, in order to show that $\mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_{n,\Gamma_d}}(\Omega))$ is dense in

$\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$, it suffices to prove that there exists $\widetilde{C} > 0$ such that

$$\|(\lambda I - A_0)\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \geq \lambda \widetilde{C} \|\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_{n,\Gamma_d}}(\Omega)) \text{ and for all } \lambda > 0, \tag{2.4.50}$$

and that the operator $I - A_0$ is onto.

Let $\theta_0 \in (\pi/2, \pi)$ be the angle determined in Proposition 2.4.7. Then, since $(0, \infty) \subset \Sigma_{\theta_0}$, Proposition 2.4.7 implies that

$$\|(\lambda I - A_0)^{-1}\mathbf{F}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \leq \frac{C}{\lambda} \|\mathbf{F}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \quad \text{for all } \mathbf{F} \in \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega) \text{ and for all } \lambda > 0, \tag{2.4.51}$$

where $C > 0$ is the constant appearing in (2.4.41). Then, using this estimate with $\mathbf{F} = (\lambda I - A_0)\mathbf{v}$, where $\mathbf{v} \in \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_{n,\Gamma_d}}(\Omega))$, we obtain that

$$\|\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} = \|(\lambda I - A_0)^{-1}(\lambda I - A_0)\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \leq \frac{C}{\lambda} \|(\lambda I - A_0)\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)}, \tag{2.4.52}$$

from where we deduce estimate (2.4.50), that is,

$$\|(\lambda I - A_0)\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)} \geq \widetilde{C}\lambda \|\mathbf{v}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)},$$

with $\widetilde{C} = 1/C$. On the other hand, the surjectivity of the operator $I - A_0$ follows from the fact that $I - A_0$ is an isomorphism from $\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega))$ into $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and from $\mathbf{V}^1_{\Gamma_d}(\Omega)$ into $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ (this follows from the Lax-Milgram theorem). This completes the proof. □

In the following theorem we establish the analyticity of the underlying semigroup associated to the Stokes operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$.

Theorem 2.4.2. *The unbounded operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.*

Proof. Thanks to [EN06, Theorem 4.6, p. 95], it suffices to show that A_0 is sectorial and densely defined in $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$. These two properties are established in Propositions 2.4.7 and 2.4.8, respectively. □

2.4.3 Expression of the pressure

In this subsection, when $h = 0$, we rewrite the pressure π in (2.1.1) in terms of $\mathbf{w}$, $\mathbf{F}$ and $\mathbf{g}$.

Let us first observe that, formally, the pressure π in system (2.1.1) is the solution of the elliptic equation

$$\Delta \pi = \operatorname{div} \mathbf{F} \ \text{ in } \Omega, \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{F} \cdot \mathbf{n} + \nu \Delta \mathbf{w} \cdot \mathbf{n} \ \text{ on } \Gamma_d, \quad \pi = 2\nu \varepsilon(\mathbf{w})\mathbf{n} \cdot \mathbf{n} \ \text{ on } \Gamma_n. \tag{2.4.53}$$

We write π in the form $\pi = q + \rho$, where q is the solution to (2.4.21) with $s = -\frac{1}{2} + \alpha$ and ρ formally satisfies the elliptic equation

$$\Delta \rho = 0 \ \text{ in } \Omega, \quad \frac{\partial \rho}{\partial \mathbf{n}} = \nu \Delta \mathbf{w} \cdot \mathbf{n} \ \text{ on } \Gamma_d, \quad \rho = 2\nu \varepsilon(\mathbf{w})\mathbf{n} \cdot \mathbf{n} \ \text{ on } \Gamma_n. \tag{2.4.54}$$

From Lemma 2.4.6 it follows that the solution q of system (2.4.21) belongs to $H^{\frac{1}{2}+\alpha,1}(\Omega)$ when $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$.

Following [FNR19], in order to define the variational problem satisfied by $\rho = \pi - q$, we introduce the problem

$$\begin{aligned} &\text{Find } \rho \in L^2(\Omega) \text{ such that} \\ &\int_\Omega \rho\zeta\, dx = 2\nu\langle \varepsilon(\mathbf{w}), \nabla^2\chi\rangle_{\mathbf{H}^{\frac{1}{2}-\alpha},\mathbf{H}^{-\frac{1}{2}+\alpha}} - 2\nu\int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi\, dx, \end{aligned} \tag{2.4.55}$$

for all $\zeta \in L^2(\Omega)$, where χ is solution to (2.4.19). In this definition we use the fact that $\mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and that the solution χ to (2.4.19) belongs to $H^{\frac{3}{2}+\alpha}(\Omega)$, for all $\alpha \in (0, \alpha^*)$ if $\zeta \in L^2(\Omega)$. Since ∇ is linear and continuous from $L^2(\Omega)$ into $\mathbf{H}^{-1}(\Omega)$ and from $H^1(\Omega)$ into $\mathbf{L}^2(\Omega)$, it is also continuous from $H^{\frac{1}{2}+\alpha}(\Omega)$ into $\mathbf{H}^{-1/2+\alpha}(\Omega)$. Thus, if $\nabla\chi$ belongs to $\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)$, $\nabla^2\chi$ is well defined as a function belonging to $(\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))^2$, and problem (2.4.55) is meaningful.

Lemma 2.4.8. *The variational problem (2.4.55) admits a unique solution $\rho \in L^2(\Omega)$.*

Proof. Let us introduce the linear functional $L : L^2(\Omega) \to \mathbb{R}$ given by

$$L(\xi) = 2\nu\langle \varepsilon(\mathbf{w}), \nabla^2\chi\rangle_{\mathbf{H}^{\frac{1}{2}-\alpha},\mathbf{H}^{-\frac{1}{2}+\alpha}} - 2\nu\int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi,$$

where χ is solution to (2.4.19) with source term $\zeta \in L^2(\Omega)$. Let us notice that

$$\begin{aligned} 2\nu\langle \varepsilon(\mathbf{w}), \nabla^2\chi\rangle_{\mathbf{H}^{\frac{1}{2}-\alpha},\mathbf{H}^{-\frac{1}{2}+\alpha}} &\le C\|\varepsilon(\mathbf{w})\|_{\mathbf{H}^{\frac{1}{2}-\alpha}}\|\nabla^2\chi\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}} \\ &\le C\|\mathbf{w}\|_{\mathbf{H}^{\frac{3}{2}+\alpha}}\|\zeta\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} -2\nu\int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi &\le C\|\varepsilon(\mathbf{w})\|_{\mathbf{L}^2(\Gamma_d)}\|\nabla\chi\|_{\mathbf{L}^2(\Gamma_d)} \\ &\le C\|\mathbf{w}\|_{\mathbf{H}^{\frac{3}{2}+\alpha}}\|\zeta\|_{L^2}. \end{aligned}$$

Thus, $L \in \mathcal{L}(L^2(\Omega), \mathbb{R})$. Then, the result follows from the Riesz representation Theorem. $\square$

Remark 5. By taking the supremum over all ζ such that $\|\zeta\|_{H^{-\frac{1}{2}+\alpha}(\Omega)} = 1$ in (2.4.55), we can prove that ρ belongs to $H^{\frac{1}{2}-\alpha}(\Omega)$. But we cannot prove that ρ belongs to $H^{\frac{1}{2}+\alpha}(\Omega)$. This result will be obtained as a consequence of the first statement in Lemma 2.4.9.

Let us introduce the operator N_v defined by

$$N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega), L^2(\Omega)), \quad N_v\mathbf{w} = \rho, \tag{2.4.56}$$

where ρ is the solution to system (2.4.55), and the operator N_p defined by

$$N_p \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega), H^{\frac{1}{2}+\alpha,1}(\Omega)), \quad N_p\mathbf{F} = q, \tag{2.4.57}$$

where q is the solution to system (2.4.21).

Lemma 2.4.9. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Let us assume that $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $h = 0$ and $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$. Let $(\mathbf{w}, \pi) \in \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ be the variational solution to system (2.1.1). Then,*

$$\pi = q + \rho,$$

where q is the solution to (2.4.21) with $s = -\frac{1}{2} + \alpha$ and ρ is the solution to the variational problem (2.4.55). As a consequence, ρ belongs to $H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ and

$$\|\rho\|_{H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)} \leq C\left(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_d)}\right).$$

Proof. We first prove the following identity

$$-\left\langle \operatorname{div}\sigma(\mathbf{w},\pi), \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}} = -\int_\Omega \pi\zeta + \int_\Omega \rho\,\zeta\,dx, \tag{2.4.58}$$

for all $\zeta \in L^2(\Omega)$, where χ is the solution of (2.4.19). Let $(\mathbf{w}_k)_k$ be a sequence in $\mathbf{H}^2(\Omega)$ converging to $\mathbf{w}$ in $\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and let $(\pi_k)_k$ be a sequence in $H^1(\Omega)$ converging to π in $H^{\frac{1}{2}+\alpha}(\Omega)$. Since div, the divergence operator, belongs to $\mathcal{L}(\mathbf{L}^2(\Omega), H^{-1}(\Omega))$ and to $\mathcal{L}(\mathbf{H}^1(\Omega), L^2(\Omega))$, by interpolation it also belongs to $\mathcal{L}(\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega), H^{-\frac{1}{2}+\alpha}(\Omega))$. Thus we have

$$\begin{aligned} &-\left\langle \operatorname{div}\sigma(\mathbf{w}_k,\pi_k), \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}} \\ &\qquad \xrightarrow[k\to+\infty]{} -\left\langle \operatorname{div}\sigma(\mathbf{w},\pi), \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}}. \end{aligned} \tag{2.4.59}$$

Using that $\sigma(\mathbf{w},\pi)\mathbf{n} = 0$ on Γ_n, we have

$$\begin{aligned} &-\left\langle \operatorname{div}\sigma(\mathbf{w}_k,\pi_k), \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}} \\ &= -\int_\Omega \operatorname{div}\sigma(\mathbf{w}_k,\pi_k)\cdot\nabla\chi \\ &= \left\langle \nabla^2\chi, \sigma(\mathbf{w}_k,\pi_k)\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}} - \int_\Gamma \sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi \\ &= -\int_\Omega \pi_k\Delta\chi + 2\nu\left\langle \nabla^2\chi, \varepsilon(\mathbf{w}_k)\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha},\mathbf{H}^{\frac{1}{2}-\alpha}} \\ &\quad - 2\nu\int_{\Gamma_d} \varepsilon(\mathbf{w}_k)\mathbf{n}\cdot\nabla\chi - \int_{\Gamma_n} \sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi \\ &= -\int_\Omega \pi_k\zeta + \int_\Omega N_v\mathbf{w}_k\zeta - \int_{\Gamma_n} \sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi \\ \xrightarrow[k\to+\infty]{} &-\int_\Omega \pi\zeta + \int_\Omega N_v\mathbf{w}\zeta - \int_{\Gamma_n} \sigma(\mathbf{w},\pi)\mathbf{n}\cdot\nabla\chi = -\int_\Omega \pi\zeta + \int_\Omega N_v\mathbf{w}\zeta. \end{aligned} \tag{2.4.60}$$

The identity (2.4.58) follows from (2.4.60) and (2.4.59).

On the other hand, we recall that the solution q to (2.4.21) with $s = -\frac{1}{2} + \alpha$ satisfies

$$\left\langle \mathbf{F}, \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} = -\int_\Omega q\,\zeta \tag{2.4.61}$$

together with the estimate (see Lemma 2.4.6)

$$\|q\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} \leq C\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)}. \tag{2.4.62}$$

Finally, since

$$-\left\langle \operatorname{div}\sigma(\mathbf{w},\pi), \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} = \left\langle \mathbf{F}, \nabla\chi\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)}, \tag{2.4.63}$$

substituting (2.4.58) and (2.4.61) into (2.4.63) yields

$$-\int_\Omega \pi\zeta + \int_\Omega N_v\mathbf{w}\zeta = -\int_\Omega N_p\mathbf{F}\zeta.$$

Thus, $\pi = \rho + q$.

The regularity of ρ follows from those of π and q (see estimate (2.4.62)). $\square$

2.4.4 System reformulated as an operator equation

Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Let us introduce the lifting operators $D \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Gamma_d), \mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega))$ and $D_p \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Gamma_d), H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega))$ defined by:

$$(D\mathbf{g}, D_p\mathbf{g}) = (\mathbf{w}, \pi), \tag{2.4.64}$$

where $(\mathbf{w}, \pi)$ is the solution of (2.1.1) when $\mathbf{F} = 0$ and $h = 0$.

Theorem 2.4.3. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Assume that $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$ and $h = 0$. A pair $(\mathbf{w}, \pi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega) \times H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)$ is a variational solution of (2.1.1) if and only if $P\mathbf{w}$, $(I-P)\mathbf{w}$, and π are solutions to the following system*

$$\begin{cases} -A_0 P\mathbf{w} + A_0 P D\mathbf{g} = P\mathbf{F}, \\ (I-P)\mathbf{w} = (I-P)D\mathbf{g}, \quad \pi = N_v\mathbf{w} + N_p\mathbf{F}, \end{cases} \tag{2.4.65}$$

where the operators N_v and N_p are introduced in (2.4.56) and (2.4.57), respectively.

Proof. Let $(\mathbf{w}, \pi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega) \times H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)$ be the solution of (2.1.1) given by Theorem 2.3.2. We set $\mathbf{w} = \widehat{\mathbf{w}} + D\mathbf{g}$ and $\pi = \widehat{\pi} + D_p\mathbf{g}$, where the couple $(\widehat{\mathbf{w}}, \widehat{\pi})$ satisfies

$$\begin{cases} -\operatorname{div}\sigma(\widehat{\mathbf{w}}, \widehat{\pi}) = \mathbf{F}, \ \ \operatorname{div}\widehat{\mathbf{w}} = 0 \ \text{ in } \Omega, \\ \widehat{\mathbf{w}} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\widehat{\mathbf{w}}, \widehat{\pi})\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{2.4.66}$$

Then, we have $\widehat{\mathbf{w}} \in \mathcal{D}(A_0)$, $-A_0\widehat{\mathbf{w}} = P\mathbf{F}$, and $(I-P)\mathbf{w} = (I-P)\widehat{\mathbf{w}} + (I-P)D\mathbf{g} = (I-P)D\mathbf{g}$, because $P\widehat{\mathbf{w}} = \widehat{\mathbf{w}}$. Since $\widehat{\mathbf{w}} = \mathbf{w} - D\mathbf{g}$, we obtain $-A_0\widehat{\mathbf{w}} = -A_0P\widehat{\mathbf{w}} = -A_0P(\mathbf{w} - D\mathbf{g})$, from which we deduce $(2.4.65)_1$. The expression of the pressure π is obtained in Lemma 2.4.9.

Let us now prove the converse of the statement. Since the solution of system (2.1.1) is also a solution for the operator equation (2.4.65), it only remains to show the uniqueness to (2.4.65). Let us first notice that $(A_0\mathbf{w}, \mathbf{w})_{\mathbf{L}^2} = 0$ implies that $\mathbf{w} = 0$. Then, if $-A_0P\mathbf{w} = 0$ and $(I-P)\mathbf{w} = 0$, we deduce that $\mathbf{w} = 0$. This completes the proof. $\square$

Appendix A: Study of the Stokes resolvent system

Given λ in the sector $\Sigma_{\theta_0} \setminus \{0\}$, with $\theta_0 \in (\pi/2, \pi)$ as in Theorem **??**, we consider the system

$$\begin{cases} \lambda \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, p) = \mathbf{F} \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{u}, p)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{2.4.67}$$

Theorem 2.4.4. *Let $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$. For all $\mathbf{F} \in H^{-1}_{\Gamma_d}(\Omega; \mathbb{C})^2$, system (2.4.67) admits a unique solution $(\mathbf{u}, p) \in H^1(\Omega; \mathbb{C})^2 \times L^2(\Omega; \mathbb{C})$. Moreover, there exists $C > 0$ such that*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \|\mathbf{F}\|_{\mathbf{H}^{-1}_{\Gamma_d}(\Omega)}. \tag{2.4.68}$$

Proof. We first introduce the Hilbert spaces

$$\mathbf{V}_{\mathbb{C}} = \Big\{ \mathbf{v} \in H^1_{\Gamma_d}(\Omega; \mathbb{C})^2 \mid \operatorname{div} \mathbf{v} = 0 \ \text{ in } \Omega \Big\} \ \text{ and } \ \mathbf{V}_{\mathbb{R}} = \Big\{ \mathbf{v} \in H^1_{\Gamma_d}(\Omega; \mathbb{R})^2 \mid \operatorname{div} \mathbf{v} = 0 \ \text{ in } \Omega \Big\}.$$

We then associate the following variational formulation to the problem (2.4.67) in the space $\mathbf{V}_{\mathbb{C}}$: Find $\mathbf{v} \in \mathbf{V}_{\mathbb{C}}$ such that

$$a(\mathbf{v}, \overline{\boldsymbol{\varphi}}) = \langle \mathbf{f}, \overline{\boldsymbol{\varphi}} \rangle_{\mathbf{H}^{-1}_{\Gamma_d}, \mathbf{H}^1_{\Gamma_d}} \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{V}_{\mathbb{C}}, \tag{2.4.69}$$

where

$$a(\mathbf{v}, \overline{\boldsymbol{\varphi}}) = \lambda \int_{\Omega} \mathbf{v} \cdot \overline{\boldsymbol{\varphi}} + 2\nu \int_{\Omega} \varepsilon(\mathbf{v}) : \varepsilon(\overline{\boldsymbol{\varphi}}). \tag{2.4.70}$$

To show that there is a unique $\mathbf{v} \in \mathbf{V}_{\mathbb{C}}$ satisfying (2.4.69), we will use the complex version of the Lax-Milgram Lemma (see [EG21, Lemma 25.2, p. 15]). For the continuous sesqulinear form a over $\mathbf{V}_{\mathbb{C}} \times \mathbf{V}_{\mathbb{C}}$, let us verify the coercivity on $\mathbf{V}_{\mathbb{C}}$. Let $\mathbf{v} \in \mathbf{V}_{\mathbb{C}}$. We distinguish two cases for $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$:

- $\underline{\Re\lambda \geq 0}$. If $\Re\lambda \geq 0$, then

$$\Re\left(a(\mathbf{v}, \mathbf{v})\right) = \Re\lambda \int_{\Omega} |\mathbf{v}|^2 + 2\nu \int_{\Omega} |\varepsilon(\mathbf{v})|^2 \geq C \|\mathbf{v}\|^2_{\mathbf{H}^1_{\Gamma_d}},$$

and therefore the coercivity is verified.

- $\underline{\Re\lambda < 0}$. Assume that $\Re\lambda < 0$. In this case, after taking $\zeta = -i \operatorname{sgn}(\Im\lambda) \dfrac{\overline{\lambda}}{|\lambda|}$ in [EG21, Lemma 25.2, p. 15], we get

$$\Re(\zeta a(\mathbf{v}, \mathbf{v})) = \frac{|\Im\lambda|}{|\lambda|} \int_{\Omega} |\varepsilon(\mathbf{v})|^2 \geq \sin(\pi - \theta_0) \|\mathbf{v}\|^2_{\mathbf{H}^1_{\Gamma_d}}.$$

Then, the variational problem (2.4.69) admits a unique solution $\mathbf{v} \in \mathbf{V}_{\mathbb{C}}$.

Let us now recover the pressure. We first introduce the Hilbert space

$$\mathbf{U}_{\mathbb{R}} = \{ \mathbf{v} \in \mathbf{H}^1_0(\Omega; \mathbb{R})^2 \mid \operatorname{div} \mathbf{v} = 0 \ \text{ in } \Omega \}.$$

Since $\mathbf{U}_{\mathbb{R}} \subset \mathbf{V}_{\mathbb{R}}$, taking $\boldsymbol{\varphi} \in \mathbf{U}_{\mathbb{R}}$ as test functions in (2.4.69), we get

$$\langle \lambda \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{F}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}, \mathbf{H}^1_0} = 0 \ \text{ for all } \boldsymbol{\varphi} \in \mathbf{U}_{\mathbb{R}}. \tag{2.4.71}$$

After taking the real and imaginary parts in identity (2.4.71) we deduce from De Rham Theorem that there exist $q_r, q_i \in L^2(\Omega;\mathbb{R})$ such that

$$\Re\left(\lambda\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{F}\right) = -\nabla q_r$$

and

$$\Im\left(\lambda\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{F}\right) = -\nabla q_i.$$

We finally define $q = q_r + iq_i \in L^2(\Omega;\mathbb{C})$. $\square$

Appendix B: Technical result adapted from Pazy's book

In this appendix, we show the following result:

Proposition 2.4.9. *Let $(X, \|\cdot\|_X)$ be a reflexive space. Assume that the unbounded linear operator $\mathcal{A}$ of domain $\mathcal{D}(\mathcal{A})$, is such that there exists $C > 0$, satisfying*

$$\|(\lambda I - \mathcal{A})x\|_X \geq C\lambda \|x\|_X \quad \text{for all } x \in \mathcal{D}(\mathcal{A}) \text{ and for all } \lambda > 0, \tag{2.4.72}$$

and the operator $I - \mathcal{A}$ is onto. Then, the following assertions hold:

(i) *$\mathcal{A}$ is closed.*

(ii) *$\lambda I - \mathcal{A}$ is onto for all $\lambda > 0$, i.e., $R(\lambda I - A) = X$ for all $\lambda > 0$.*

(iii) *The domain $\mathcal{D}(\mathcal{A})$ is dense in X.*

The three assertions stated in the proposition are established in [Paz83] in the case $C = 1$ in (2.4.72). Assertions (i) and (ii) are shown in [Paz83, Theorem 4.3 (a), p. 14-15], while assertion (iii) is proved in [Paz83, Theorem 4.6, p.16]. In the proof of the Proposition 2.4.9 presented below, we revisit the arguments presented in [Paz83].

Proof of Proposition 2.4.9.

(i) Using the fact that $I - \mathcal{A}$ is onto together with estimate (2.4.72) for $\lambda = 1$, we deduce that $(I - \mathcal{A})^{-1}$ is bounded and thus closed. Then, $I - \mathcal{A}$ is closed an therefore also $\mathcal{A}$ is closed.

(ii) Let us consider the set

$$\Lambda = \{\lambda \mid 0 < \lambda < \infty \quad \text{and} \quad R(\lambda I - A) = X\}.$$

Let $\lambda \in \Lambda$. By (2.4.72), λ belongs to the resolvent set $\rho(A)$. Then, since $\rho(A)$ is open, a neighborhood of λ is in $\rho(A)$. The intersection of this neighborhood with the real line is in Λ and therefore Λ is open. On the other hand, let $\lambda_n \in \Lambda$ such that $\lambda_n \longrightarrow \lambda > 0$. For every $y \in X$, there exists $x_n \in \mathcal{D}(\mathcal{A})$ such that

$$\lambda_n x_n - \mathcal{A}x_n = y. \tag{2.4.73}$$

From estimate (2.4.72) it follows that $\|x_n\|_X \leq \lambda_n^{-1}\|y\|_X \leq K$ for some $K > 0$. Now,

$$\begin{aligned}\lambda_m \|x_n - x_m\|_X &\leq \|\lambda_m(x_n - x_m) - \mathcal{A}(x_n - x_m)\|_X \\ &\leq |\lambda_n - \lambda_m| \|x_n\|_X \leq K|\lambda_n - \lambda_m|.\end{aligned}$$

Therefore $(x_n)_n$ is a Cauchy sequence. Let $x_n \longrightarrow x$. Then, by (2.4.73), $\mathcal{A}x_n \longrightarrow \lambda x - y$. From ($i$), $\mathcal{A}$ is closed, and then, $x \in \mathcal{D}(\mathcal{A})$ and $\lambda x - \mathcal{A}x = y$. Therefore, $R(\lambda I - \mathcal{A}) = X$ and $\lambda \in \Lambda$. Thus, Λ is also closed in $]0, \infty[$. Finally, since $1 \in \Lambda$ and $]0, \infty[$ is connected, $\Lambda =]0, \infty[$.

(iii) Let $x^* \in X'$ be such that $\langle x^*, x \rangle_{X',X} = 0$ for every $x \in \mathcal{D}(\mathcal{A})$. We will prove that $x^* = 0$ on X. Since $I - \mathcal{A}$ is onto, it suffices to show that $\langle x^*, x - \mathcal{A}x \rangle_{X',X} = 0$ for every $x \in \mathcal{D}(\mathcal{A})$, which is equivalent to $\langle x^*, Ax \rangle_{X',X} = 0$ for every $x \in \mathcal{D}(\mathcal{A})$. Let $x \in \mathcal{D}(\mathcal{A})$. Then, by ($ii$) there exists $x_n \in \mathcal{D}(\mathcal{A})$ such that $x = x_n - (1/n)\mathcal{A}x_n$. Since $\mathcal{A}x_n = n(x_n - x) \in \mathcal{D}(\mathcal{A})$, $x_n \in \mathcal{D}(\mathcal{A}^2)$ and $\mathcal{A}x = \mathcal{A}x_n - (1/n)\mathcal{A}^2 x_n$ or $\mathcal{A}x_n = (I - (1/n)\mathcal{A})^{-1}\mathcal{A}x$. From (2.4.72), it follows that $\|(I - (1/n)\mathcal{A})^{-1}\|_X \leq 1/C$ and therefore $\|\mathcal{A}x_n\|_X \leq (1/C)\|\mathcal{A}x\|_X$. We also have that $\|x_n - x\|_X \leq (1/n)\|\mathcal{A}x_n\|_X \leq (1/Cn)\|\mathcal{A}x\|_X$ and therefore $x_n \longrightarrow x$. Then, since $\|\mathcal{A}x_n\| \leq K$ and X is reflexive, there exists a subsequence $(\mathcal{A}x_{n_k})_k$ of $(\mathcal{A}x_n)_n$ such that $\mathcal{A}x_{n_k} \longrightarrow y$ weakly. On the other hand, since by (i) $\mathcal{A}$ is closed, then $y = \mathcal{A}x$. Finally, we have

$$\langle x^*, \mathcal{A}x_{n_k} \rangle_{X',X} = n_k \langle x^*, x_{n_k} - x \rangle_{X',X} = 0. \tag{2.4.74}$$

Letting $n_k \longrightarrow +\infty$ in (2.4.74), we obtain $\langle x^*, Ax\rangle_{X',X} = 0$. This holds for every $x \in \mathcal{D}(\mathcal{A})$ and therefore $x^* = 0$ and thus, $\mathcal{D}(\mathcal{A})$ is dense in X. This completes the proof.

Appendix C: Justification of the identity (2.4.55) (formal computations)

We consider the system

$$\Delta\rho = 0 \text{ in } \Omega,\ \frac{\partial\rho}{\partial\mathbf{n}} = \nu\Delta\mathbf{w}\cdot\mathbf{n} \text{ on } \Gamma_d,\ \ \rho = 2\nu\varepsilon(\mathbf{w})\mathbf{n}\cdot\mathbf{n} \text{ on } \Gamma_n, \tag{2.4.75}$$

where $(\mathbf{w},\pi)$ is the solution of the system (2.1.1).

The aim of this appendix is to present the formal computations to get the identity (2.4.55) in the definition of the *very weak* notion of solution for (2.4.75), as introduced in (2.4.55).

Let us consider auxiliar the system

$$\Delta\chi = \zeta \text{ in } \Omega,\ \frac{\partial\chi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_d,\ \ \chi = 0 \text{ on } \Gamma_n. \tag{2.4.76}$$

We claim that

$$\int_\Omega \rho\zeta = 2\nu\int_\Omega \varepsilon(\mathbf{w}) : \nabla^2\chi - 2\nu\int_{\Gamma_d}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi. \tag{2.4.77}$$

After multiplying the first equation in (2.4.75) by $\nabla\chi$, and then performing some integration by parts, we get

$$\begin{aligned}\int_\Omega \rho\zeta &= -\int_\Gamma \frac{\partial\rho}{\partial\mathbf{n}}\chi + \int_\Gamma \rho\frac{\partial\chi}{\partial\mathbf{n}}\\ &= -\nu\int_{\Gamma_d}\Delta\mathbf{w}\cdot\mathbf{n}\chi + 2\nu\int_{\Gamma_n}\varepsilon(\mathbf{w})\mathbf{n}\cdot\mathbf{n}\frac{\partial\chi}{\partial\mathbf{n}}\\ &= -\nu\int_{\Gamma_d}\Delta\mathbf{w}\cdot\mathbf{n}\chi + 2\nu\int_{\Gamma_n}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi,\end{aligned} \tag{2.4.78}$$

from where we deduce that

$$\int_\Omega \rho\zeta = -\nu\int_{\Gamma_d}\Delta\mathbf{w}\cdot\mathbf{n}\chi + 2\nu\int_{\Gamma_n}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi. \tag{2.4.79}$$

In the chain of equalities (2.4.78), to go from the second line to the third, we use the fact that $\chi = 0$ on Γ_n, which allows us to deduce that $\nabla\chi = \dfrac{\partial\chi}{\partial\mathbf{n}}\mathbf{n}$ on Γ_n.

After multiplying by $\nabla\chi$ the first equation in (2.1.1), we get

$$\int_\Omega \mathbf{F}\cdot\nabla\chi - \int_\Omega \nabla\pi\cdot\nabla\chi = -\int_\Omega \operatorname{div}(2\nu\varepsilon(\mathbf{w}))\cdot\nabla\chi. \tag{2.4.80}$$

Let us first notice that after performing some integration by parts on the second term of the identity (2.4.80), we obtain

$$\begin{aligned}-\int_\Omega \operatorname{div}(2\nu\varepsilon(\mathbf{w}))\cdot\nabla\chi &= 2\nu\int_\Omega\varepsilon(\mathbf{w}) : \nabla^2\chi - 2\nu\int_\Gamma\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi\\ &= 2\nu\int_\Omega\varepsilon(\mathbf{w}) : \nabla^2\chi - 2\nu\int_{\Gamma_d}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi - 2\nu\int_{\Gamma_n}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi.\end{aligned} \tag{2.4.81}$$

On the other hand, since $\operatorname{div}\mathbf{w} = 0$ in Ω, we can rewrite the second term of the identity (2.4.80) as follows:

$$\begin{aligned} -\int_\Omega \operatorname{div}(2\nu\varepsilon(\mathbf{w}))\cdot\nabla\chi &= -\nu\int_\Omega \Delta\mathbf{w}\cdot\nabla\chi \\ &= \nu\int_\Omega \operatorname{div}(\Delta\mathbf{w})\cdot\chi - \nu\int_\Gamma \Delta\mathbf{w}\chi\cdot\mathbf{n} \\ &= -\nu\int_{\Gamma_d} \Delta\mathbf{w}\chi\cdot\mathbf{n}. \end{aligned} \tag{2.4.82}$$

After combining identities (2.4.81) and (2.4.82), we get

$$-\nu\int_{\Gamma_d} \Delta\mathbf{w}\cdot\mathbf{n}\chi = 2\nu\int_\Omega \varepsilon(\mathbf{w}) : \nabla^2\chi - 2\nu\int_{\Gamma_d}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi - 2\nu\int_{\Gamma_n}\varepsilon(\mathbf{w})\mathbf{n}\cdot\nabla\chi. \tag{2.4.83}$$

Finally, replacing (2.4.83) in (2.4.79), we obtain the identity (2.4.77).

Chapter 3

Existence of a local-in-time strong solution of the fluid-structure system

Abstract of the current chapter

In this chapter, we analyse a system modeling the interaction between the incompressible Navier-Stokes equations in a 2D rectangular domain with mixed boundary conditions, and a structure governed by a damped Euler-Bernoulli beam equation. The structure, which is assumed to be clamped at one end and free at the other one, is immersed in the domain occupied by the fluid. We prove the existence of a local in time strong solution.

Outline of the current chapter

3.1 Introduction **69**
- 3.1.1 Description of the model 69
- 3.1.2 Discussion 71
- 3.1.3 Outline 72

3.2 Notation and statement of the main result **72**
- 3.2.1 Notation 73
- 3.2.2 System in the reference configuration 74
- 3.2.3 Statement of the main result 77

3.3 System rewritten using the semigroup formulation **79**
- 3.3.1 Steady Stokes system 79
- 3.3.2 Euler-Bernoulli beam equation 82
- 3.3.3 Coupled linear fluid-structure system 83
- 3.3.4 Analyticity of the semigroup generated by $(\mathcal{A}, \mathcal{D}(A))$ on $\mathbf{Z}$ 85

3.4 Existence and regularity results for the non-homogeneous linear system **88**

3.5 Estimates of nonlinear terms **96**
- 3.5.1 Auxiliary results 96
- 3.5.2 Estimates of nonlinear terms 100

3.6 Proof of Theorem 3.2.1 **105**

Appendix A: Expression of the function H and simplification in the modelling **107**
- Discussion about the term J_3 109

Appendix B: Change of variables **111**
- B.1 General change of variable formula 111
- B.2 Nonlinear terms coming from the geometric transformation (3.2.9) 111

Appendix C: Auxiliar results **112**
C.1 Study of Stokes system with irregular divergence data (case 1) . . 112
C.2 Study of the Stokes system with irregular divergence data (case 2) 113
C.3 Auxiliar Lemma 113
Appendix D: Proof of estimates (3.5.17) **and** (3.5.18) **114**
D.1 Proof of estimate (3.5.17) 115
D.2 Proof of estimate (3.5.18) 115

3.1 Introduction

3.1.1 Description of the model

In this paper, we study a fluid-structure interaction problem coupling an incompressible and viscous Newtonian fluid modeled by the incompressible Navier-Stokes equations, and an elastic solid $\mathcal{S}_e$ obeying a damped Euler-Bernoulli beam equation. The elastic solid, which is immersed within the fluid, is clamped at one end to a rigid cylinder $\mathcal{S}_r$ and free at the other end. See Figure 3.1.

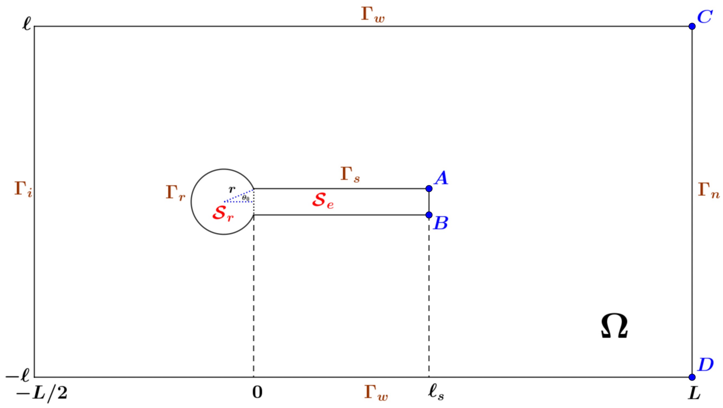


Figure 3.1 – Reference configuration.

The reference configuration Ω of the fluid domain is given by

$$\Omega = ([-L/2, L] \times [-\ell, \ell]) \setminus \mathcal{S},$$

where $\mathcal{S} = \mathcal{S}_r \cup \mathcal{S}_e$. The boundary Γ of Ω is divided into

$$\Gamma = \Gamma_i \cup \Gamma_r \cup \Gamma_s \cup \Gamma_w \cup \Gamma_n,$$

where

$$\begin{aligned}
&\Gamma_i = \{-L/2\} \times [-\ell, \ell],\\
&\Gamma_r = \{(r(\cos(\theta) - \cos(\theta_0)), r\sin(\theta)) \mid \theta \in [\theta_0, 2\pi - \theta_0]\}, \quad r > 0, \theta_0 \in (0, \pi/2),\\
&\Gamma_s = \Gamma_s^- \cup \Gamma_s^+ \cup \Gamma_s^\ell,\\
&\Gamma_w = [-L/2, L] \times \{-\ell\} \cup [-L/2, L] \times \{\ell\},\\
&\Gamma_n = \{L\} \times [-\ell, \ell],
\end{aligned}$$

with $\Gamma_s^- = [0, \ell_s] \times \{-e\}$, $\Gamma_s^+ = [0, \ell_s] \times \{e\}$ and $\Gamma_s^\ell = \{\ell_s\} \times [-e, e]$. We also set $\Gamma_d = \Gamma \setminus \Gamma_n$. See Figure 3.1.

When the dynamic is active, the interaction between the fluid and the structure involves a deformation of the geometry as is illustrated in Figure 3.2.

Let $T > 0$. For a given function η defined from $(0, T) \times (0, \ell_s)$ to $\mathbb{R}$ that describes the displacement of the centerline of the beam, we denote by $\Omega_{\eta(t)}$ the fluid domain at time t and by $\Gamma_{\eta(t)} = \Gamma_{\eta(t)}^- \cup \Gamma_{\eta(t)}^+ \cup \Gamma_{\eta(t)}^\ell$ the fluid-structure interface, where $\Gamma_{\eta(t)}^-$ and $\Gamma_{\eta(t)}^+$ represent the bottom and the top of the structure respectively, and $\Gamma_{\eta(t)}^\ell$ the lateral part, see Figure 3.2. Here, we assume that the elastic part $\mathcal{S}_e$ of the structure is completely described by the displacement η of the centerline. More precisely, $\Gamma_{\eta(t)}^-$, $\Gamma_{\eta(t)}^+$ and $\Gamma_{\eta(t)}^\ell$ are given by

$$\Gamma_{\eta(t)}^- = \{(x_1, \eta(t, x_1) - e) |\ x_1 \in [0, \ell_s]\}, \ \Gamma_{\eta(t)}^+ = \{(x_1, \eta(t, x_1) + e) |\ x_1 \in [0, \ell_s]\}$$

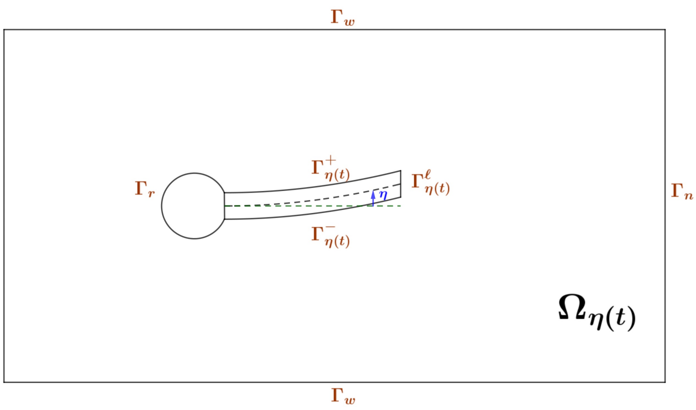


Figure 3.2 – Physical domain. The green-dashed lines denotes the reference centerline.

and

$$\Gamma^\ell_{\eta(t)} = \{(\ell_s, x_2) \mid x_2 = (1-\lambda)(-e+\eta(t,\ell_s)) + \lambda(e+\eta(t,\ell_s))), \ \ \lambda \in [0,1]\}.$$

For $0 < T \leq \infty$, we set

$$\begin{aligned}
&Q^T_\eta = \bigcup_{t\in(0,T)} \Big(\{t\}\times\Omega_{\eta(t)}\Big), \ \Sigma^T_\eta = \bigcup_{t\in(0,T)} \Big(\{t\}\times\Gamma_{\eta(t)}\Big),\\
&Q^T = (0,T)\times\Omega, \ \ \Sigma^T_s = (0,T)\times\Gamma_s,\\
&\Sigma^T_i = (0,T)\times\Gamma_i, \ \ \Sigma^T_w = (0,T)\times\Gamma_w,\\
&\Sigma^T_r = (0,T)\times\Gamma_r, \ \ \Sigma^T_n = (0,T)\times\Gamma_n.
\end{aligned}$$

In the model, the fluid velocity $\mathbf{u}$ and the pressure p satisfy the incompressible Navier-Stokes equations

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q^T_\eta, \tag{3.1.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \ \text{ in } Q^T_\eta, \tag{3.1.1b}$$

$$\mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma^T_i, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma^T_w, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma^T_r, \tag{3.1.1c}$$

$$\mathbf{u} = \eta_t\vec{\mathbf{e}}_2 \ \text{ on } \Sigma^T_\eta, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma^T_n, \tag{3.1.1d}$$

$$\mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega, \tag{3.1.1e}$$

where the fluid stress tensor $\sigma(\mathbf{u},p)$ is given by

$$\sigma(\mathbf{u},p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

with $\nu > 0$ denoting the fluid viscosity.

We assume that the displacement of the centerline η is modeled by the damped Euler-Bernoulli beam equation

$$\partial^2_t\eta + \alpha\Delta^2_s\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) \text{ in } (0,T)\times(0,\ell_s), \tag{3.1.2a}$$

$$\eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ on } (0,T)\times\{0\}, \tag{3.1.2b}$$

$$\partial^2_{x_1}\eta = 0 \ \text{ and } \ \partial^3_{x_1}\eta = 0 \ \text{ on } (0,T)\times\{\ell_s\}, \tag{3.1.2c}$$

$$\eta(0) = 0 \ \text{ and } \ \partial_t\eta(0) = \eta^0_2 \ \text{ in } (0,\ell_s), \tag{3.1.2d}$$

where the parameters $\alpha > 0$ and $\gamma > 0$ are constants relative to the nature of the structure. Throughout the paper, we assume that $\Delta^2_s = \frac{\partial^4}{\partial x^4}$ with domain $\mathcal{D}(\Delta^2_s) = H^4_{\{0,\ell_s\}}(0,\ell_s)$ stands for the bi-Laplace operator on $(0,\ell_s)$ and we set $B = (\Delta^2_s)^{\frac{1}{2}}$ with domain $\mathcal{D}(B) = H^2_{\{0\}}(0,\ell_s)$ (see

Section 3.2.1). With this choice of B, system (3.1.2) is known as a damped Euler-Bernoulli beam equation with a structural type damping (see [CR82]). The source term H on the right-hand side of the structure equation is given by

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2, \tag{3.1.3}$$

where

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x_1,\eta(t,x_1)\pm e), p(t,x_1,\eta(t,x_1)\pm e)),$$

and $\mathbf{n}^+_{\eta(t)}$ (resp. $\mathbf{n}^-_{\eta(t)}$) is the unit normal to $\Gamma^+_{\eta(t)}$ (resp. $\Gamma^-_{\eta(t)}$) exterior to $\Omega_{\eta(t)}$. This expression for H is derived from the system's energy identity. See Appendix A.

The equations (3.1.1) and (3.1.2) have to be completed by the so-called kinematic-dynamic interface coupling conditions. The matching kinematic condition has already been included in system (3.1.1) and is given by

$$\mathbf{u} = \eta_t\vec{\mathbf{e}}_2 \ \text{ on } \Sigma^T_\eta.$$

The dynamic coupling condition is encapsulated in the function H defined in (5.1.2) that appears on the right-hand side of equation (3.1.2). This term describes the force exerted by the fluid on $\Gamma^+_{\eta(t)} \cup \Gamma^-_{\eta(t)}$.

3.1.2 Discussion

Following the article of Quarteroni et al. [QTV00], there have been many works studying the coupling between the incompressible Navier-Stokes equations with beam or plate equations. Let us present a non-exhaustive brief review below.

Concerning the study of the existence of weak solutions, to the best of our knowledge, the first result in this direction was done by Chambolle et al. [CDEG05], where they studied a system coupling the 3D incompressible Navier-Stokes equations and a 2D damped plate equation. This result was later extended by Grandmont [Gra08] to the undamped case. In both works, which are also valid in the 2D/1D setting, the existence of weak solutions holds as long as the structure does not touch the fixed part of the fluid boundary. Recently, Casanova et al. [CGH21], show in the 2D/1D undamped setting, the existence of global in time weak solutions regardless of a possible contact between the structure and the fixed part of the fluid domain.

Regarding the study of strong solutions, to the best of our knowledge, the first result was obtained by Da Veiga [Bei04] in the case where the dynamic of the structure is governed by a damped Euler-Bernoulli beam equation. There, the author shows a local-in-time existence result under the assumption of a smallness condition on the data. Later, this result was improved by Lequeurre [Leq11], who proves the same local in time existence result showed by da Veiga, but without the assumption of smallness condition on the data. The study of the undamped case is done in Badra et al. [BT19], where the authors show the existence of a local in time strong solution.

The model studied in the present paper considers a structure immersed in a fluid, with one end free, but with the restriction that its displacement is only tranversal. The main contribution of this work is the proof of the existence of a local-in-time strong solution to system (3.1.1)-(3.1.2), see Theorem 3.2.1. Let us now describe the strategy used in the proof, along with the main difficulties encountered in the analysis.

Since the fluid domain evolves over time, we rewrite the system in a fixed reference domain by introducing an appropriate change of variables. After performing the change of variables, we must deal with the presence of additional nonlinearities in the new system written in the reference configuration. To address the nonlinear problem in this framework, we employ a classical approach. First, we associate to the nonlinear problem a linear one involving nonhomogeneous source terms, where the latter represent the nonlinearities. Then, after solving the linear problem, we use the Banach fixed point theorem to treat the nonlinear problem. The crucial points are the establishement of the well-posedness of the linear problem and the Lipschitz estimates used in the fixed point argument. Indeed, both steps are interconnected in the sense that the regularity imposed on the inhomogeneous data of the linear problem must be appropriate to obtain the estimates of the nonlinear terms in the fixed point argument. This key aspect reduces to studying the spatial regularity of the fluid velocity and pressure in the linear inhomogeneous problem. More precisely, we have to study the regularity of $\mathbf{w}$ and pressure π of the stationary Stokes system

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{w},\pi)=\mathbf{F} \ \text{ in } \Omega, \ \ \operatorname{div}\mathbf{w}=h \ \text{ in } \Omega, \\ \mathbf{w}=\mathbf{g} \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n}=0 \ \text{ on } \Gamma_n, \end{cases} \tag{3.1.4}$$

where $\mathbf{F}$, h and $\mathbf{g}$ are stationary data. Here, $\mathbf{F}$ and h play the role of the nonlinear terms obtained after rewriting the system in the reference configuration. We first highlight that the domain Ω has reentrant corners at A and B (see Figure 3.1). Secondly, we emphasize the presence of the junctions points between Dirichlet and Neumann boundary conditions at vertices C and D. To the best of our knowledge, a first regularity result that take into account the two issues mentioned above is the one stated by Maz'ya and Rossmann in [MR10, Theorem 9.4.5]. This result allows us to deduce that the solution $(\mathbf{w},\pi)$ of the system (3.1.4) satisfies the following regularity:

$$\begin{aligned} &(\mathbf{w},\pi) \text{ belongs to the weighted Sobolev space } \mathbf{H}^2_\delta(\Omega)\times H^1_\delta(\Omega) \\ &\text{when } \mathbf{F}\in\mathbf{L}^2(\Omega), \ h\in H^1(\Omega) \text{ and } \mathbf{g}\in\mathbf{H}^{\frac{3}{2}}(\Gamma_d), \end{aligned} \tag{3.1.5}$$

for all $\delta\in(\delta^*,1)$, where $\delta^*\in(0,1/2)$. The weighted Sobolev spaces $\mathbf{H}^2_\delta(\Omega)$, $H^1_\delta(\Omega)$ are introduced in (3.2.2).

Next, the crucial question to answer is determine whether all the nonlinear terms encoded in $\mathbf{F}$ and h belong to $\mathbf{L}^2(\Omega)$ and $H^1(\Omega)$, respectively. In this regard, based on [FNR19, Lemma 6.3], the answer to this question is negative. In fact, there are certain terms in $\mathbf{F}$ for which we are unable to establish the $\mathbf{L}^2$-estimate, for instance, the term $\mathbf{w}_{xx}\eta$. Indeed, the cited lemma above states that if $\mathbf{w}\in\mathbf{H}^2_\delta(\Omega)$ and $\eta\in H^1(-L/2,L)$ with $\eta(0)=\eta(\ell_s)=0$, then $\mathbf{w}_{xx}\eta$ belongs to $\mathbf{L}^2(\Omega)$. The core argument of the proof of this result relies on the assumption $\eta(0)=\eta(\ell_s)=0$. However, in our setting, this is not generally the case, as $\eta(\ell_s)\neq 0$. For this reason, we must seek an alternative regularity result. In particular, an adapted version of (3.1.5) is provided by the authors in Section 2.3 of Chapter 2, in terms of heterogeneous Sobolev spaces. The heterogeneous Sobolev spaces are introduced in 3.2.1.

3.1.3 Outline

The remainder of this paper is organized as follows. In Section 3.2, after introducing the notation of the different functional spaces, we perform a change of variable to rewrite system (3.1.1)-(3.1.2) in the reference configuration and state the main result of the paper. Section 3.3 is devoted to rewrite the nonhomogeneous linear system as an operator equation. We rewrite the stationary Stokes system in Subsection 3.3.1 and the structure equation in Subsection 3.3.2. The coupled linear fluid-structure system is rewritten as an evolution equation in Subsection 3.3.3. In Subsection 3.3.4 is studied the analyticity of the underlying semigroup. An existence and uniqueness result for the nonhomogeneous linear system is presented in section 3.4. In Section 3.5, we estimate the nonlinear terms. In Section 3.6, we show the main result of the paper by using a fixed point argument. Finally, some technical results are collected in Appendix.

3.2 Notation and statement of the main result

Throughout this paper, the letter C will denote a constant (independent of T) which may vary from line to line.

3.2.1 Notation

Usual and weighted Sobolev spaces

We set $\mathbf{L}^2(\Omega) = L^2(\Omega; \mathbb{R}^2)$ and $\mathbf{H}^s(\Omega) = H^s(\Omega; \mathbb{R}^2)$ for $s > 0$. We also introduce the following functional spaces:

$$\begin{aligned}
&\mathbf{H}^s_{\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{H}^s(\Omega) \mid \mathbf{u} = 0 \text{ on } \Gamma_d\} \ \text{ for } s > 1/2,\\
&\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \ \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_d\},\\
&\mathbf{V}^1_{\Gamma_d}(\Omega) = \mathbf{H}^1_{\Gamma_d}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega),\\
&H^1_{\{0\}}(0,\ell_s) = \{\mu \in H^1(0,\ell_s) \mid \mu(0) = 0\},\\
&H^2_{\{0\}}(0,\ell_s) = \{\mu \in H^2(0,\ell_s) \mid \mu(0) = \partial_{x_1}\mu(0) = 0\},\\
&H^3_{\{0,\ell_s\}}(0,\ell_s) = \{\mu \in H^3(0,\ell_s) \cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s) = 0\},\\
&H^4_{\{0,\ell_s\}}(0,\ell_s) = \{\mu \in H^4(0,\ell_s) \cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s) = \partial^3_{x_1}\mu(\ell_s) = 0\},\\
&H^{2,1}_{\{0\}}((0,T) \times (0,\ell_s)) = L^2(0,T; H^2_{\{0\}}(0,\ell_s)) \cap H^1(0,T; L^2(0,\ell_s)),\\
&H^{4,2}_{\{0,\ell_s\}}((0,T) \times (0,\ell_s)) = L^2(0,T; H^4_{\{0,\ell_s\}}(0,\ell_s)) \cap H^2(0,T; L^2(0,\ell_s)).
\end{aligned}$$

All of the previous spaces are endowed with the natural norms.

We introduce the space of the inflow conditions

$$\mathbf{H}(\Gamma_i) = \Big\{ \mathbf{g} = (g_1, g_2) \mid g_2 = 0 \ \text{ and } \ g_1 \in H^{\frac{3}{2}}(\Gamma_i) \cap H^1_0(\Gamma_i) \Big\}, \tag{3.2.1}$$

equipped with the norm $(g_1, g_2) \mapsto \|g_1\|_{H^{\frac{3}{2}}(\Gamma_i)}$.

If we identify $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ with its dual and, if $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ denotes the dual of $\mathbf{V}^1_{\Gamma_d}(\Omega)$, we have

$$\mathbf{V}^1_{\Gamma_d}(\Omega) \hookrightarrow \mathbf{V}^0_{n,\Gamma_d}(\Omega) \hookrightarrow \mathbf{V}^{-1}_{\Gamma_d}(\Omega)$$

with dense and continuous embeddings. For $0 < s < 1/2$, we introduce the intermediate spaces

$$\mathbf{H}^s_{\Gamma_d}(\Omega) = [\mathbf{L}^2(\Omega), \mathbf{H}^1_{\Gamma_d}(\Omega)]_s.$$

For the definition of interpolation spaces by using the complex interpolation method see e.g. [Tar07] or [Tri78]. The dual of $\mathbf{H}^s_{\Gamma_d}(\Omega)$ is denoted by $\mathbf{H}^{-s}_{\Gamma_d}(\Omega)$.

Let us denote by $\mathcal{J}$ the set of vertices of Γ. For $\beta > 0$, we introduce the norms

$$\begin{aligned}
\|\mathbf{w}\|_{\mathbf{H}^2_\beta} &:= \left(\sum_{|k|=0}^{2} \sum_{i=1}^{2} \int_\Omega \Big(\prod_{J \in \mathcal{J}} r_J^{2\beta} \Big) |\partial^k w_i|^2 dx \right)^{1/2}, \quad \mathbf{w} \in C^\infty(\overline{\Omega}; \mathbb{R}^2)\\
\|p\|_{H^1_\beta} &:= \left(\sum_{|k|=0}^{1} \int_\Omega \Big(\prod_{J \in \mathcal{J}} r_J^{2\beta} \Big) |\partial^k p|^2 dx \right)^{1/2}, \quad p \in C^\infty(\overline{\Omega}; \mathbb{R})
\end{aligned} \tag{3.2.2}$$

where r_J stands for the distance to a junction point $J \in \mathcal{J}$, $k = (k_1, k_2) \in \mathbb{N}^2$ denotes a two-index with length $|k| = k_1 + k_2$, ∂_k denotes the corresponding partial differential operator and $\mathbf{w} = (w_1, w_2)$. We denote by $\mathbf{H}^2_\beta(\Omega; \mathbb{R}^2)$ (respectively, $H^1_\beta(\Omega)$) the closure of $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ (respectively, $C^\infty(\overline{\Omega})$) in the norm $\|\cdot\|_{\mathbf{H}^2_\beta(\Omega)}$ (respectively, $\|\cdot\|_{H^1_\beta(\Omega)}$).

Throughout this chapter, we will use the following regularity exponents introduced in Subsection 2.2.2 of Chapter 2:

$$\alpha^* \in (0, 1/2) \quad \text{and} \quad \delta^* \in (0, 1/2). \tag{3.2.3}$$

Heterogeneous Sobolev spaces

Let $\varepsilon > 0$. We introduce the cut-off function $\Psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $0 \leq \Psi \leq 1$,

$$\Psi = 1 \text{ on } (-L/2, \ell_s + \varepsilon/2) \times (-\ell, \ell) \text{ and } \Psi = 0 \text{ on } (\ell_s + \varepsilon, L) \times (-\ell, \ell). \tag{3.2.4}$$

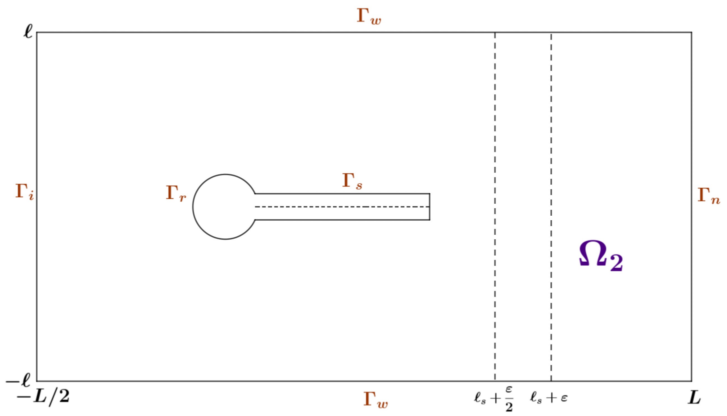


Figure 3.3 – Decomposition of the domain Ω.

We also define $\Omega_2 = (\ell_s + \varepsilon/2, L) \times (-\ell, \ell)$. We now introduce the heterogeneous Sobolev spaces

$$\begin{aligned}
\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) &= \left\{ \mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1(\Omega) \right\}, \\
H_\delta^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H_\delta^1(\Omega) \right\}, \\
\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) &= \left\{ \mathbf{v} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{v} \in \mathbf{H}_\delta^2(\Omega) \right\},
\end{aligned} \tag{3.2.5}$$

which are respectively endowed with the norms

$$\begin{aligned}
\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}} &:= \left(\|\mathbf{F}\|^2_{\mathbf{H}^{-\frac{1}{2}+\alpha}} + \|(1-\Psi)\mathbf{F}\|^2_{\mathbf{L}^2} \right)^{1/2}, \\
\|p\|_{H^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H^1} \right)^{1/2}, \\
\|p\|_{H_\delta^{\frac{1}{2}+\alpha,1}} &:= \left(\|p\|^2_{H^{\frac{1}{2}+\alpha}} + \|(1-\Psi)p\|^2_{H_\delta^1} \right)^{1/2}, \\
\|\mathbf{u}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}} &:= \left(\|\mathbf{u}\|^2_{\mathbf{H}^{\frac{3}{2}+\alpha}} + \|(1-\Psi)\mathbf{u}\|^2_{\mathbf{H}_\delta^2} \right)^{1/2}.
\end{aligned}$$

3.2.2 System in the reference configuration

This subsection is devoted to rewriting the system (3.1.1)-(3.1.2) in the reference configuration. The spatial variable in the physical domain is denoted by $x = (x_1, x_2)$, while the spatial variable in the reference configuration will be denoted by $z = (z_1, z_2)$.
For $r > 1/2$, we denote by $\gamma_s^\pm$ the trace operators belonging to $\mathcal{L}(\mathbf{H}^r(\Omega), H^{r-\frac{1}{2}}(\Gamma_s^\pm))$, which are

defined by

$$(\gamma_s^+ q)(x_1) = q(x_1, e) \text{ and } (\gamma_s^- q)(x_1) = q(x_1, -e) \text{ for all } x_1 \in (0, \ell_s). \tag{3.2.6}$$

Let us now introduce an appropriate extension of any function defined on $[0, \ell_s]$ to $[-L/2, L]$. We set $\eta \mapsto \mathcal{E}\eta$, where

$$\mathcal{E}\eta(x_1) = \begin{cases} 0 & \text{if } x_1 \in [-L/2, 0], \\ \eta(x_1) & \text{if } x_1 \in [0, \ell_s], \\ (3\eta(2\ell_s - x_1) - 2\eta(3\ell_s - 2x_1))\, \theta(x_1) & \text{if } x_1 \in [\ell_s, L], \end{cases} \tag{3.2.7}$$

where $\theta \in C^\infty([\ell_s, L])$ is a nonnegative function with values in $[0, 1]$, which is equal to 1 in $[\ell_s, \ell_s + \varepsilon/4]$ and to 0 in $[\ell_s + \varepsilon/2, L]$, for some $0 < \varepsilon < (L - \ell_s)/2$. The following proposition is a direct consequence of definition of $\mathcal{E}$.

Proposition 3.2.1. *For all $T > 0$ and $\eta \in H^{4,2}_{\{0,\ell_s\}}((0, T) \times (0, \ell_s))$, the following assertions are satisfied.*

(*i*) *For all $0 < a_0 < 1/2$,*

$$\mathcal{E}\eta \in L^2(0, T; H^{2+a_0}(-L/2, L)) \cap H^2(0, T; L^2(-L/2, L)).$$

and

$$\mathcal{E}\eta \in H^{\frac{1+2a_0-2\tau}{2+a_a}}(0, T; H^{\frac{3}{2}+\tau}(-L/2, L)), \quad \textit{for all } \tau \in (0, 3a_0/4).$$

In particular, $(\mathcal{E}\eta)(t, \cdot)$ is a map of class $\mathcal{C}^1$.

(*ii*) *For all $0 < \varepsilon_0 < L$,*

$$(\mathcal{E}\eta)|_{(\varepsilon_0, L)} \in L^2(0, T; H^4(\varepsilon_0, L)).$$

(*iii*) $(\mathcal{E}\eta)|_{[\ell_s+\varepsilon/2, L]} \equiv 0$.

For a given $\eta \in H^{4,2}_{\{0,\ell_s\}}((0, T) \times (0, \ell_s))$, we set

$$\eta^+(t, \cdot, e) := (\mathcal{E}\eta)(t, \cdot) \text{ on } [-L/2, L] \quad \text{and} \quad \eta^-(t, \cdot, -e) := (\mathcal{E}\eta)(t, \cdot) \text{ on } [-L/2, L].$$

We now introduce two C^∞ functions χ^+ and χ^-, with values in $[0, 1]$, such that $\chi^+(z_2) = 1$ in $[\frac{e}{2}, \ell]$ and $\chi^+(z_2) = 0$ in $[-\ell, \frac{e}{3}]$, $\chi^-(z_2) = 1$ in $[-\ell, -\frac{e}{2}]$ and $\chi^-(z_2) = 0$ in $[-\frac{e}{3}, \ell]$.

For $T > 0$, we introduce the set

$$E(0, T) = \Big\{\eta \in H^{4,2}_{\{0,\ell_s\}}((0, T) \times (0, \ell_s)) \text{ such that} \\ \min\{\ell - e + \eta^\pm_\chi(t, z) \mid (t, z) \in [0, T] \times \overline{\Omega}\} \geq (\ell - e)/2\Big\}. \tag{3.2.8}$$

where

$$\eta^\pm_\chi(t, z) := (\mp\chi^\pm + (\ell \mp z_2)\partial_{z_2}\chi^\pm)\, \eta^\pm(t, z_1), \text{ with } (t, z) \in [0, T] \times \overline{\Omega}.$$

For η belonging to $E(0, T)$, we consider the map $X(t, \cdot) : \Omega \to \Omega_{\eta(t)}$ defined by $X(t, z) = x = (x_1, x_2)$, where

$$\begin{cases} x_1 = z_1, \\ x_2 = \chi^\pm(z_2) \dfrac{z_2(\ell - e \mp \eta^\pm(t, z_1)) + \ell\eta^\pm(t, z_1)}{\ell - e} + (1 - \chi^\pm(z_2))z_2, \end{cases} \tag{3.2.9}$$

for all $z = (z_1, z_2) \in \Omega$. Given $0 < \varepsilon_0 < L$ introduced in Proposition 3.2.1, let us set

$$\begin{aligned} \mathcal{O}_R &:= ([\varepsilon_0, \ell_s] \times ([-\ell, -e] \cup [e, \ell])) \cup ([\ell_s, L] \times [-\ell, \ell]) \,, \\ \mathcal{O}_L &:= (\Omega \setminus \mathcal{O}_R) \cup ([\varepsilon_0, \varepsilon_1] \times ([-\ell, -e] \cup [e, \ell])) \,. \end{aligned} \tag{3.2.10}$$

See Figure 3.4.

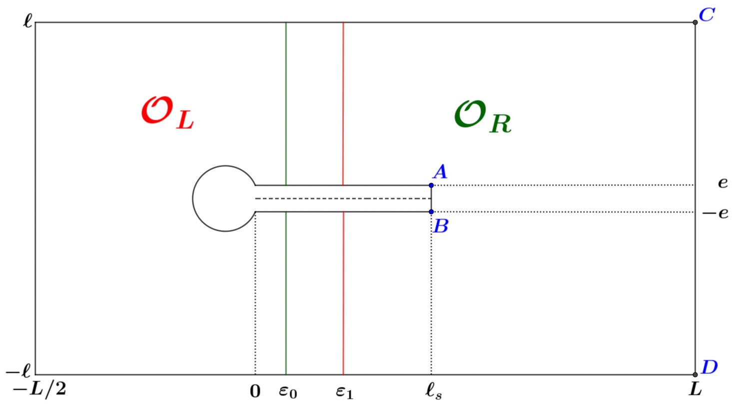


Figure 3.4 – Decomposition of the domain Ω.

The following Proposition is a consequence of the definition of the mapping X given in (3.2.9) and Proposition 3.2.1.

Proposition 3.2.2. *Let $T > 0$. For all $\eta \in E(0,T)$, the mapping X defined in (3.2.9) satisfies:*

(i) $X(0, \Omega) = \Omega$.

(ii) For all $t \in [0,T]$, we have that $X(t, \Gamma_i) = \Gamma_i$, $X(t, \Gamma_w) = \Gamma_w$, $X(t, \Gamma_r) = \Gamma_r$, $X(t, \Gamma_s) = \Gamma_{\eta(t)}$ and $X(t, \Gamma_n) = \Gamma_n$.

(iii) $X \in L^2(0,T; \mathbf{H}^{2+a_0}(\mathcal{O}_L))$ *for all* $0 < a_0 < 1/2$, *and* $X \in L^2(0,T; \mathbf{H}^4(\mathcal{O}_R))$.

(iv) $X \in H^2(0,T; \mathbf{L}^2(\Omega))$.

(v) $X(t,\cdot)$ is a $\mathcal{C}^1-$diffeomorphism from Ω onto $\Omega_{\eta(t)}$.

We will denote by Y the inverse of X. We also set $J(t,z) = (J^{i,j})_{1 \le i,j \le 2} = (\nabla X)^{-1}(t,z)$ for all $(t,z) \in (0,T) \times \Omega$. Let us notice that for X defined in (3.2.9)

$$\det(J) = \frac{\ell - e}{\ell - e + \eta_X^{\pm}(t,z)}. \tag{3.2.11}$$

In order to transform the system (3.1.1)-(3.1.2) in the reference configuration, we introduce the change of unknowns

$$\widehat{\mathbf{u}}(t,z) = \mathbf{u}(t, X(t,z)) \ \text{ and } \ \widehat{p}(t,z) = p(t, X(t,z)), \tag{3.2.12}$$

for all $(t,z) \in (0,T) \times \Omega$. We get that $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ satisfies the system

$$\partial_t \widehat{\mathbf{u}} - \operatorname{div} \sigma(\widehat{\mathbf{u}}, \widehat{p}) = \widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}, \widehat{p}, \eta) \ \text{ in } Q^T, \tag{3.2.13a}$$

$$\operatorname{div} \widehat{\mathbf{u}} = \operatorname{div} \widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}, \eta) \ \text{ in } Q^T, \tag{3.2.13b}$$

$$\widehat{\mathbf{u}} = \mathbf{g}^i \ \text{ on } \Sigma_i^T, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_w^T, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_r^T, \tag{3.2.13c}$$

$$\widehat{\mathbf{u}} = \eta_t \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \ \ \sigma(\widehat{\mathbf{u}}, \widehat{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^T, \tag{3.2.13d}$$

$$\widehat{\mathbf{u}}(0) = \mathbf{u}_0 \ \text{ in } \Omega, \tag{3.2.13e}$$

$$\partial_t^2 \eta + \alpha \Delta_s^2 \eta + \gamma (\Delta_s^2)^{\frac{1}{2}} \partial_t \eta = -\gamma_s^+ \widehat{p} + \gamma_s^- \widehat{p} + \widehat{F}_s(\widehat{\mathbf{u}}, \widehat{p}, \eta) \ \text{ in } (0,T) \times (0, \ell_s), \tag{3.2.13f}$$

$$\eta = 0 \ \text{ and } \ \partial_{z_1} \eta = 0 \ \text{ on } (0,T) \times \{0\}, \tag{3.2.13g}$$

$$\partial^2_{z_1}\eta = 0 \ \text{ and } \ \partial^3_{z_1}\eta = 0 \ \text{ on } (0,T)\times\{\ell_s\}, \tag{3.2.13h}$$
$$\eta(0) = 0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } (0,\ell_s), \tag{3.2.13i}$$

where the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ and $\widehat{F}_s$ are given by

$$\begin{aligned}\widehat{\mathbf{F}}_{f,i}(\widehat{\mathbf{u}},\widehat{p},\eta) &= -(\widehat{\mathbf{u}}-\partial_t X)\cdot J^\top\nabla\widehat{u}_i + \nu\sum_{j,k,\ell}\frac{\partial^2\widehat{u}_i}{\partial z_k\partial z_\ell}(J^{k,j}J^{l,j}-\delta_{k,j}\delta_{\ell,j})\\ &+\nu\sum_{j,k,\ell}\frac{\partial\widehat{u}_i}{\partial z_k}\frac{\partial}{\partial z_\ell}(J^{k,j})J^{\ell,j} + \Big((I_{\mathbb{R}^2}-J^\top)\nabla q\Big)_i - \nu\left(\nabla\left(\nabla\widehat{\mathbf{u}} : (I_{\mathbb{R}^2}-\mathrm{cof}(\nabla X))\right)\right)_i,\end{aligned} \tag{3.2.14}$$

$i = 1,2$,

$$\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}},\eta) = (I_{\mathbb{R}^2}-\mathrm{cof}(\nabla X)^\top)\widehat{\mathbf{u}} \tag{3.2.15}$$

and

$$\begin{aligned}\widehat{F}_s(\widehat{\mathbf{u}},\eta) = &-\nu\gamma_s^+\left(\eta_x\left[\sum_k\frac{\partial\widehat{u}_1}{\partial z_k}\frac{\partial Y_k}{\partial x_2}+\sum_k\frac{\partial\widehat{u}_2}{\partial z_k}\frac{\partial Y_k}{\partial x_1}\right]\right)+2\nu\gamma_s^+\sum_k\frac{\partial\widehat{u}_2}{\partial z_k}\frac{\partial Y_k}{\partial x_2}\\ &+\nu\gamma_s^-\left(\eta_x\left[\sum_k\frac{\partial\widehat{u}_1}{\partial z_k}\frac{\partial Y_k}{\partial x_2}+\sum_k\frac{\partial\widehat{u}_2}{\partial z_k}\frac{\partial Y_k}{\partial x_1}\right]\right)-2\nu\gamma_s^-\sum_k\frac{\partial\widehat{u}_2}{\partial z_k}\frac{\partial Y_k}{\partial x_2}.\end{aligned} \tag{3.2.16}$$

The explicit change of variable for the map $X(t,\cdot)$ given in (3.2.9), is presented in the Appendix B.

3.2.3 Statement of the main result

Following [MRR20], we introduce the definition of Sobolev spaces in the time-dependent domain $\Omega_{\eta(t)}$.

Definition 3.2.1. *Let $\alpha\in(0,\alpha^*)$, $\delta\in(\delta^*,1)$ and $T>0$. For any $\eta\in E(0,T)$, we say that $\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,1}(\Omega_{\eta(\cdot)}))$ (resp. $H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega_{\eta(\cdot)}))$ if there exists X belonging to $L^2(0,T;\mathbf{H}^{2+a_0}(\mathcal{O}_L))\cap L^2(0,T;\mathbf{H}^4(\mathcal{O}_R))\cap H^2(0,T;\mathbf{L}^2(\Omega))$, with $0<a_0<1/2$, such that for all $t\in[0,T]$, $X(t,\cdot)$ is a $\mathcal{C}^1-$diffeomorphism from Ω onto $\Omega_{\eta(t)}$, and when $\widehat{\mathbf{u}}$ defined by*

$$\widehat{\mathbf{u}}(t,z) = \mathbf{u}(t,X(t,z)),\quad \textit{for all } (t,z)\in[0,T]\times\Omega,$$

belongs to $L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))$ (resp. $H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$). Similarly, we will say that p belongs to $L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega_{\eta(\cdot)}))$ when $\widehat{p}$, defined by

$$\widehat{p}(t,z) = p(t,X(t,z)),\quad \textit{for all } (t,z)\in[0,T]\times\Omega,$$

belongs to $L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$.

We are interested in solutions $(\mathbf{u},p,\eta)$ to system (3.1.1)-(3.1.2) satisfying

$$\begin{aligned}&\mathbf{u}\in L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega_{\eta(\cdot)}))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega_{\eta(\cdot)})),\\ &p\in L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega_{\eta(\cdot)})),\\ &\eta\in L^2(0,T;H^4_{\{0,\ell_s\}}(0,\ell_s))\cap H^2(0,T;L^2(0,\ell_s)).\end{aligned} \tag{3.2.17}$$

Definition 3.2.2. *We say that the triple $(\mathbf{u},p,\eta)$ is a strong solution to system (3.1.1)-(3.1.2) over the time interval $(0,T)$, when it satisfies (3.2.17), equations (3.1.1a)-(3.1.1b) in the sense of distributions in Q_η^T, equation (3.1.2a) in the sense of distributions in $(0,T)\times(0,\ell_s)$, equa-*

tions (3.1.1c)-(3.1.1d)-(3.1.2b)-(3.1.2c) *in the sense of traces, and the initial conditions stated in* (3.1.1e) *and* (3.1.2d).

Before presenting the main result of the paper, let us introduce some additional notation. Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. We introduce the space

$$\begin{aligned} \mathcal{Z}_T = \Big(L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \Big) \\ \times L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)), \end{aligned} \tag{3.2.18}$$

equipped with the norm

$$\begin{aligned} \|(\mathbf{u},p,\eta)\|_{\mathcal{Z}_T} = \|\mathbf{u}\|_{L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ + \|p\|_{L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\eta\|_{H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))}. \end{aligned} \tag{3.2.19}$$

We also introduce the set

$$\begin{aligned} B(T,R,\mathbf{u}_0,\eta_2^0) := \Big\{ (\widehat{\mathbf{u}},\widehat{p},\eta) \in \mathcal{Z}_T \mid \|(\widehat{\mathbf{u}},\widehat{p},\eta)\|_{\mathcal{Z}_T} \le R,\ \ \eta \in E(0,T) \\ \text{and } \widehat{\mathbf{u}}(0) = \mathbf{u}_0,\ \eta(0) = 0,\ \eta_t(0) = \eta_2^0 \Big\}. \end{aligned} \tag{3.2.20}$$

Let us now state the main results of the paper.

Theorem 3.2.1. *For all* $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H^1_{\{0\}}(0,\ell_s)$ *and* $\mathbf{g}^i \in H^1_{\{0\}}(0,1;\mathbf{H}(\Gamma_i))$ *satisfying*

$$\begin{aligned} &\mathbf{u}_0 = 0 \ \text{ on } \Gamma_i, \quad \mathbf{u}_0 = 0 \ \text{ on } \Gamma_r \cup \Gamma_w, \\ &\mathbf{u}_0 = \eta_2^0(0,\cdot)\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \operatorname{div} \mathbf{u}_0 = 0 \ \text{ in } \Omega, \end{aligned} \tag{3.2.21}$$

there exist $T \in (0,1)$ *and* $R > 0$ *such that the system* (3.2.13) *admits a unique solution* $(\widehat{\mathbf{u}},\widehat{p},\eta)$ *in* $B(T,R,\mathbf{u}_0,\eta_2^0)$. *In addition, if we set*

$$\mathbf{u}(t,x) = \widehat{\mathbf{u}}(t,X^{-1}(t,x)) \ \text{ and } \ p(t,x) = \widehat{p}(t,X^{-1}(t,x)), \ \text{ for all } x \in \Omega_{\eta(t)},\ t \in [0,T],$$

where the map $X(t,\cdot) : \Omega \longrightarrow \Omega_{\eta(t)}$ *is the one introduced in* (3.2.9), *then* $(\mathbf{u},p,\eta)$ *is a solution to system* (3.1.1)-(3.1.2).

Remark 6. Theorem 3.2.1 can be extended to the case in which the damping operator B in equation (3.1.2a) is given by $B = (\Delta_s^2)^r$, with domain $\mathcal{D}(B) = D((\Delta_s^2)^r)$, for $1/2 < r \le 1$. Throughout the proof developed in this paper, the damping operator B only plays a role in proving the analyticity of the semigroup generated by it. However, thanks to [CT89, Proposition 3.1], the analyticity of the semigroup associated to $B = (\Delta_s^2)^r$, with domain $\mathcal{D}(B) = D((\Delta_s^2)^r)$, is valid for all $r \in [1/2, 1]$. We emphasize that the case $r = 1/2$ of structural-type damping considered in this work is primarily motivated by an ongoing study on a stabilization problem, where this specific choice enables the application of the approach used in [FNR19].

3.3 System rewritten using the semigroup formulation

Let us consider the non-homogeneous linear system given by

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{F}_f \ \text{ in } Q^T, \\ \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{G}_{\mathrm{div}} \ \text{ in } Q^T, \\ \mathbf{v} = \mathbf{g}^i \ \text{ on } \Sigma_i^T, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma_w^T \cup \Sigma_r^T, \ \ \mathbf{v} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \\ \sigma(\mathbf{v}, p)\mathbf{n} = 0 \ \text{ on } \Sigma_n^T, \\ \mathbf{v}(0) = \mathbf{v}_0 \ \text{ in } \Omega, \\ \partial_t \zeta_1 = \zeta_2 \ \text{ on } (0,T) \times (0, \ell_s), \\ \partial_t \zeta_2 + \alpha \Delta_s^2 \zeta_1 + \gamma (\Delta_s^2)^{\frac{1}{2}} \zeta_2 = -\gamma_s^+ p + \gamma_s^- p + F_s \ \text{ in } (0,T) \times (0, \ell_s), \\ \zeta_1 = 0 \ \text{ and } \ \partial_{x_1} \zeta_1 = 0 \ \text{ on } (0,T) \times \{0\}, \\ \partial_{x_1}^2 \zeta_1 = 0 \ \text{ and } \ \partial_{x_1}^3 \zeta_1 = 0 \ \text{ on } (0,T) \times \{\ell_s\}, \\ \zeta_1(0) = 0 \ \text{ and } \ \zeta_2(0) = \zeta_2^0 \ \text{ in } (0, \ell_s). \end{cases} \tag{3.3.1}$$

In this section, we shall rewrite system (3.3.1) as an operator equation when $\mathbf{F}_f = \mathbf{0}$, $\mathbf{G}_{\mathrm{div}} = \mathbf{0}$ and $F_s = 0$ (see Proposition 3.3.4). First, in Subsection 3.3.1 we recall some results of the steady stationary Stokes system presented in Chapter 2. Next, in Subsection 3.3.2 we study the structure equation. The analysis of the coupled linear fluid-structure interaction system (3.3.1) is presented in Subsection 3.3.3.

3.3.1 Steady Stokes system

We consider the stationary Stokes system

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{w}, \pi) = \mathbf{F} \ \text{ in } \Omega, \ \ \operatorname{div} \mathbf{w} = h \ \text{ in } \Omega, \\ \mathbf{w} = \mathbf{g} \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w}, \pi)\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_n. \end{cases} \tag{3.3.2}$$

Let us first start recalling some useful properties stated in Chapter 2.

• **Regularity result**. In Theorems 2.3.1 and 2.3.2 of Chapter 2, we show the following existence, uniqueness and regularity results for system (3.3.2).

Theorem 3.3.1. *Let assume that $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$.*

(i) *For all $(\mathbf{F}, h, \mathbf{g}) \in \mathbf{H}_{\Gamma_d}^{-1}(\Omega) \times L^2(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\Gamma_d)$, system (3.3.2) admits a unique variational solution $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$.*

(ii) *For all $(\mathbf{F}, h, \mathbf{g}) \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) \times H^{\frac{1}{2}+\alpha,1}(\Omega) \times \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$, the variational solution $(\mathbf{w}, \pi)$ of system (3.3.2) belongs to $\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$. Moreover, there exists a constant $C_\alpha > 0$, such that*

$$\begin{aligned} &\|\mathbf{w}\|_{\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)} + \|\pi\|_{H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)} \\ &\qquad \leq C_\alpha \Big(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} + \|h\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_i)} \Big). \end{aligned} \tag{3.3.3}$$

Remark 7. The case when $h = \operatorname{div} \mathbf{G}_{\mathrm{div}}$, with $\mathbf{G}_{\mathrm{div}}$ irregular, which will be used later in the proof of Proposition 3.4.2, is treated in Appendix C.1.

We now introduce the lifting operators D and D_p defined by

$$D \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Gamma_d), \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)), \ \ D_p \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Gamma_d), H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)),$$

such that

$$(D\mathbf{g}, D_p\mathbf{g}) = (\mathbf{w}, \pi), \tag{3.3.4}$$

where $(\mathbf{w}, \pi)$ is the solution of (3.3.2) when $\mathbf{F} = 0$, $h = 0$ and $\mathbf{g} = 0$.

• **Stokes operator**. We begin recalling the following result stated in [NR15, Lemma 2.2].

Proposition 3.3.1. *We have that*

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_{n,\Gamma_0}(\Omega) \overset{\perp}{\oplus} \nabla H^1_{\Gamma_n}(\Omega), \tag{3.3.5}$$

where $H^1_{\Gamma_n}(\Omega) = \{p \in H^1(\Omega) \mid p = 0 \text{ on } \Gamma_n\}$. Moreover, the orthogonal projection P from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$ is defined by $P\mathbf{F} = \mathbf{F} - \nabla q$, where q is solution of the variational problem

$$\begin{aligned} &\text{Find } q \in H^1_{\Gamma_n}(\Omega) \text{ such that} \\ &\int_\Omega \nabla q \cdot \nabla \phi \, dx = \int_\Omega \mathbf{F} \cdot \nabla \phi \, dx, \quad \forall \phi \in H^1_{\Gamma_n}(\Omega). \end{aligned} \tag{3.3.6}$$

The following proposition is proved Corollary 2.4.1.

Proposition 3.3.2. *The operator $P \in \mathcal{L}(\mathbf{L}^2(\Omega))$ defined by*

$$P\mathbf{F} = \mathbf{F} - \nabla q, \tag{3.3.7}$$

where q is the solution to (3.3.6), is also continuous from $\mathbf{H}^s(\Omega)$ into itself for all $s \in [0, 1/2+\alpha^*)$. Furthermore, $P \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))$ and $P \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$.

For $0 < s < 1/2$, we introduce the space

$$\mathbf{V}^s_{n,\Gamma_d}(\Omega) = \mathbf{H}^s_{\Gamma_d}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega), \tag{3.3.8}$$

which is equipped with the Sobolev $\mathbf{H}^s_{\Gamma_d}-$norm, and we define $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$ as the dual of $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$ with $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ as pivot space, equipped with the dual norm of $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$.

Remark 8. Let $s \in (0, 1/2)$. Thanks to Lemma 2.4.5 of Chapter 2, we can identify the dual space of $\mathbf{V}^s_{n,\Gamma_d}(\Omega)$, denoted by $\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega)$, with the closed subspace $\overline{\mathbf{V}^0_{n,\Gamma_d}(\Omega)}^{\|\cdot\|_{\mathbf{H}^{-s}_{\Gamma_d}}}$ of $\mathbf{H}^{-s}_{\Gamma_d}(\Omega)$. We will use this identification throughout this chapter.

The Stokes operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is defined by

$$\begin{aligned} \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)) = \Big\{ &\mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \cap \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists \pi \in H^{\frac{1}{2}+\alpha}(\Omega) \text{ such that } \operatorname{div} \sigma(\mathbf{w}, \pi) \in \mathbf{L}^2(\Omega) \\ &\text{and } \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \text{ on } \Gamma_n \Big\}, \\ &A_0\mathbf{w} = P \operatorname{div} \sigma(\mathbf{w}, \pi). \end{aligned}$$

For all $\theta \in (\pi/2, \pi)$, we define the sector Σ_θ by

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \theta\}.$$

The following theorem is proved in Theorem 2.4.1 of Chapter 2.

Theorem 3.3.2. *There exist $\theta_0 \in (\pi/2, \pi)$ and $C > 0$ such that*

$$\|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(\mathbf{V}^0_{n,\Gamma_d}(\Omega))} \le \frac{C}{|\lambda|}, \quad \textit{for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\}.$$

In particular, the unbounded operator $(A_0, \mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$.

Since the domain Ω satisfies the assumption $(H1)-(H4)$ stated in Subsection 2.2.1 of Chapter 2, the same proof of Lemma 2.4.7 (ii) can be used to show the following result.

Lemma 3.3.1. *For all $s\in(0,1/2)$,*

$$[\mathbf{V}^0_{n,\Gamma_d}(\Omega),\mathbf{V}^1_{\Gamma_d}(\Omega)]_s=\mathbf{V}^s_{n,\Gamma_d}(\Omega)\ \ \text{and}\ \ [\mathbf{V}^0_{n,\Gamma_d}(\Omega),\mathbf{V}^{-1}_{\Gamma_d}(\Omega)]_s=\mathbf{V}^{-s}_{n,\Gamma_d}(\Omega).$$

Let us now introduce the Stokes operator A_0 on $\mathbf{V}^{-\frac12+\alpha,0}_{n,\Gamma_d}(\Omega)$. In the same way as in Chapter 2, using the preceding lemma and the fact that A_0 is an isomorphism from $\mathcal{D}(A_0;\mathbf{V}^0_{n,\Gamma_d}(\Omega))$ into $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and from $\mathcal{D}(A_0;\mathbf{V}^{-1}_{\Gamma_d}(\Omega))$ into $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ (this follows from the Lax-Milgram theorem), we deduce that A_0 is also an isomorphism from

$$\begin{aligned}&\mathcal{D}(A_0;\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega)):=[\mathcal{D}(A_0;\mathbf{V}^0_{n,\Gamma_d}(\Omega)),\mathcal{D}(A_0;\mathbf{V}^{-1}_{\Gamma_d}(\Omega))]_{\frac12-\alpha}\\&\text{into }[\mathbf{V}^0_{n,\Gamma_d}(\Omega),\mathbf{V}^{-1}_{\Gamma_d}(\Omega)]_{\frac12-\alpha}=\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega).\end{aligned}\tag{3.3.9}$$

But, since $\mathcal{D}(A_0;\mathbf{V}^{-1}_{\Gamma_d}(\Omega))=\mathbf{V}^1_{\Gamma_d}(\Omega)$,

$$\mathcal{D}(A_0;\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega))=[\mathcal{D}(A_0;\mathbf{V}^0_{n,\Gamma_d}(\Omega)),\mathbf{V}^1_{\Gamma_d}(\Omega)]_{\frac12-\alpha}.$$

We now recall the analyticity of the underlying semigroup associated to the Stokes operator $(A_0,\mathcal{D}(A_0;\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega)))$ on $\mathbf{V}^{-\frac12+\alpha,0}_{n,\Gamma_d}(\Omega)$ established in Theorem 2.4.2 of Chapter 2.

Theorem 3.3.3. *The unbounded operator $(A_0,\mathcal{D}(A_0;\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^{-\frac12+\alpha}_{n,\Gamma_d}(\Omega)$.*

- **Expression of the pressure**. Let us assume that $h=0$. Formally, the pressure π in system (3.3.2) is the solution of the elliptic equation

$$\Delta\pi=\operatorname{div}\mathbf{F}\ \text{ in }\Omega,\ \ \frac{\partial\pi}{\partial\mathbf{n}}=\mathbf{F}\cdot\mathbf{n}+\nu\Delta\mathbf{w}\cdot\mathbf{n}\ \text{ on }\Gamma_d,\ \ \pi=2\nu\varepsilon(\mathbf{w})\mathbf{n}\cdot\mathbf{n}\ \text{ on }\Gamma_n.\tag{3.3.10}$$

We write π in the form $\pi=q+\rho$, where q is the formal solution of the elliptic equation

$$\Delta q=\operatorname{div}\mathbf{F}\ \text{ in }\Omega,\ \ \frac{\partial q}{\partial\mathbf{n}}=\mathbf{F}\cdot\mathbf{n}\ \text{ on }\Gamma_d,\ \ q=0\ \text{ on }\Gamma_n,\tag{3.3.11}$$

and ρ satisfies the elliptic equation

$$\Delta\rho=0\ \text{ in }\Omega,\ \ \frac{\partial\rho}{\partial\mathbf{n}}=\nu\Delta\mathbf{w}\cdot\mathbf{n}\ \text{ on }\Gamma_d,\ \ \rho=2\nu\varepsilon(\mathbf{w})\mathbf{n}\cdot\mathbf{n}\ \text{ on }\Gamma_n.\tag{3.3.12}$$

Let us now introduce the operator $N_p\in\mathcal{L}(\mathbf{H}^{-\frac12+\alpha,0}(\Omega),H^{\frac12+\alpha,1}(\Omega))$ defined by

$$N_p\mathbf{F}=q,\tag{3.3.13}$$

where q is the solution to system (3.3.11). Let us also introduce the operator $N_v\in\mathcal{L}(\mathbf{H}^{\frac32+\alpha}(\Omega),L^2(\Omega))$ defined by

$$N_v\mathbf{w}=\rho,\tag{3.3.14}$$

where ρ is the solution to system (3.3.12). These operators are well-defined (see Subsection 2.4.3 in Chapter 2)

We recall the following result established in Theorem 2.4.3 of Chapter 2:

Theorem 3.3.4. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Assume that $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$ and $h = 0$. A pair $(\mathbf{w}, \pi) \in \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ is solution to (3.3.2) if and only if*

$$\begin{cases} -A_0 P\mathbf{w} + A_0 PD\mathbf{g} = P\mathbf{F}, \\ (I - P)\mathbf{w} = (I - P)D\mathbf{g}, \\ \pi = N_v \mathbf{w} + N_p \mathbf{F}. \end{cases} \tag{3.3.15}$$

3.3.2 Euler-Bernoulli beam equation

In this subsection, we consider the structure equation

$$\begin{cases} \partial_t \eta_1 = \eta_2 \ \text{ in } (0, T) \times (0, \ell_s), \\ \partial_t \eta_2 + \alpha \Delta_s^2 \eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}} \eta_2 = 0 \ \text{ in } (0, T) \times (0, \ell_s), \\ \eta_1 = 0 \ \text{ and } \ \partial_{x_1} \eta_1 = 0 \ \text{ on } (0, T) \times \{0\}, \\ \partial_{x_1}^2 \eta_1 = 0 \ \text{ and } \ \partial_{x_1}^3 \eta_1 = 0 \ \text{ on } (0, T) \times \{\ell_s\}, \\ \eta(0) = 0 \ \text{ and } \ \eta_2(0) = \eta_2^0 \ \text{ in } (0, \ell_s). \end{cases} \tag{3.3.16}$$

In order to rewrite system (3.3.16) as an operator equation, we introduce the state space

$$H_s = H^2_{\{0\}}(0, \ell_s) \times L^2(0, \ell_s), \tag{3.3.17}$$

where we recall that $H^2_{\{0\}}(0, \ell_s) = \{\mu \in H^2(0, \ell_s) \mid \mu(0) = \partial_{x_1}\mu(0) = 0\}$. The space H_s is equipped with the inner product

$$\langle (\eta_1, \eta_2), (\zeta_1, \zeta_2) \rangle_{H_s} = \int_0^{\ell_s} \left(\alpha \partial_{x_1}^2 \eta_1 \partial_{x_1}^2 \zeta_1 + \eta_2 \zeta_2 \right) dx_1. \tag{3.3.18}$$

Let us now consider the unbounded operator $\Delta_s^2 = \frac{\partial^4}{\partial x^4}$ with domain $\mathcal{D}(\Delta_s^2) = H^4_{\{0,\ell_s\}}(0, \ell_s)$. Since $(\Delta_s^2, \mathcal{D}(\Delta_s^2))$ is self-adjoint and positive in $L^2(0, \ell_s)$, we can define the operator $(\Delta_s^2)^{\frac{1}{2}}$ as in [Paz83, Section 2.6].

We now define the unbounded operator $(A_s, \mathcal{D}(A_s))$ in H_s by

$$\mathcal{D}(A_s) = \mathcal{D}(\Delta_s^2) \times H^2_{\{0\}}(0, \ell_s), \ \ A_s = \begin{pmatrix} 0 & I \\ -\alpha \Delta_s^2 & -\gamma(\Delta_s^2)^{\frac{1}{2}}. \end{pmatrix} \tag{3.3.19}$$

Theorem 3.3.5. *The unbounded operator $(A_s, \mathcal{D}(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s.*

Proof. Since the unbounded operator $(\Delta_s^2, \mathcal{D}(\Delta_s^2))$ in $L^2(0, \ell_s)$ is self-adjoint, positive, with dense domain in $L^2(0, \ell_s)$ and compact resolvent, by taking $B = (\Delta_s^2)^{1/2}$ with $\mathcal{D}(B) = H^2_{\{0\}}(0, \ell_s)$ in [CT89, Proposition 3.1], we conclude the result. □

Proposition 3.3.3. *Let us assume that $\eta_2^0 \in H^1_{\{0\}}(0, \ell_s)$. Then, system (3.3.16) admits a unique solution $(\eta_1, \eta_2) \in H^{4,2}_{\{0,\ell_s\}}((0, T) \times (0, \ell_s)) \times H^{2,1}((0, T) \times (0, \ell_s))$.*

Proof. Since $(A_s, \mathcal{D}(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s and $(0, \eta_2^0) \in [\mathcal{D}(A_s), H_s]_{1/2} = H^3_{\{0,\ell_s\}}(0, \ell_s) \times H^1_{\{0\}}(0, \ell_s)$, from [BDDM07, Theorem 3.1, Chapter 1], we deduce that the pair (η_1, η_2) belongs to $L^2(0, T; \mathcal{D}(A_s)) \cap H^1(0, T; H_s)$. Thus,

$$\eta_1 \in L^2(0, T; H^4_{\{0,\ell_s\}}(0, \ell_s)) \cap H^1(0, T; H^2_{\{0\}}(0, \ell_s))$$

and

$$\eta_2 \in L^2(0,T;H^2_{\{0\}}(0,\ell_s)) \cap H^1(0,T;L^2(0,\ell_s)).$$

This completes the proof. $\square$

3.3.3 Coupled linear fluid-structure system

In this subsection, we are going to rewrite system (3.3.1) as an evolution equation when $h = 0$ and $\mathbf{g}^i = 0$. We start by introducing the lifting operators D^v_s and D^p_s defined by

$$(D^v_s\eta_2, D^p_s\eta_2) = (\mathbf{w}, \pi), \tag{3.3.20}$$

where $(\mathbf{w}, \pi)$ is solution of system (3.3.2). We also introduce the operator $N_s \in \mathcal{L}(L^2(0,\ell_s), H^1(\Omega))$ which is defined by $N_s\kappa = q$, where q is solution of

$$\begin{cases} \Delta q = 0 \ \text{ in } \Omega, \dfrac{\partial q}{\partial \mathbf{n}} = \kappa \text{ on } \Gamma_s^+, \ \dfrac{\partial q}{\partial \mathbf{n}} = -\kappa \text{ on } \Gamma_s^-, \\ \dfrac{\partial q}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \\ q = 0 \text{ on } \Gamma_n. \end{cases} \tag{3.3.21}$$

Proposition 3.3.4. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*,1)$ and $T > 0$. Assume that $\eta_2 \in H^{2,1}((0,T)\times(0,\ell_s))$. Then, $\mathbf{v} \in L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$ and $q \in L^2(0,T;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega))$ is a solution of the system*

$$\begin{cases} \partial_t\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},q) = 0, \ \ \operatorname{div}\mathbf{v} = 0 \ \ \textit{in } Q^T, \\ \mathbf{v} = 0 \ \ \textit{on } \Sigma_i^T, \ \ \mathbf{v} = 0 \ \ \textit{on } \Sigma_r^T \cup \Sigma_w^T, \ \ \mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \ \textit{on } \Sigma_s^T, \\ \sigma(\mathbf{v},q)\mathbf{n} = 0 \ \ \textit{on } \Sigma_n^T, \\ \mathbf{v}(0) = \mathbf{v}^0 \ \ \textit{in } \Omega, \end{cases} \tag{3.3.22}$$

if and only if,

$$\begin{cases} P\mathbf{v}' = A_0P\mathbf{v} - A_0PD^v_s\eta_2, \ \ P\mathbf{v}(0) = P\mathbf{v}_0, \\ (I-P)\mathbf{v} = (I-P)D^v_s\eta_2, \ \ q = N_v\mathbf{v} + N_s\eta_{2,t}. \end{cases} \tag{3.3.23}$$

Remark 9. The solution of the operator equation (3.3.23) has to be understood in the sense of [BDDM07, Definition 3.1(v), p. 129].

Proof. From Theorem 3.3.4 we know that the pair $(\mathbf{v},q)$ belonging to $\left(L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)\right) \times L^2(0,T;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega))$ is the solution to system (3.3.22), if and only if,

$$\begin{cases} P\mathbf{v}' = A_0P\mathbf{v} - A_0PD^v_s\eta_2, \ \ P\mathbf{v}(0) = P\mathbf{v}_0, \\ (I-P)\mathbf{v} = (I-P)D^v_s\eta_2, \ \ q = N_v\mathbf{v} + N_p(-\mathbf{v}_t). \end{cases}$$

Then, according to the definition of the operator N_s (see (3.3.21)) we deduce that $N_p(-\partial_t\mathbf{v}) = N_s\partial_t\eta_2$. $\square$

We now proceed to rewrite the structure equation. Let us assume that $\mathbf{F}_f = 0$, $\mathbf{g}^i = \mathbf{0}$ and $F_s = 0$. Since the pressure is given by $q = N_v\mathbf{v} + N_s\eta_{2,t}$, we have

$$\begin{aligned} \partial_t\eta_2 + \alpha\Delta_s^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 &= -\gamma_s^+q + \gamma_s^-q \\ &= -\gamma_s^+N_v\mathbf{v} - \gamma_s^+N_s\partial_t\eta_2 + \gamma_s^-N_v\mathbf{v} + \gamma_s^-N_s\partial_t\eta_2. \end{aligned}$$

Thus,

$$\Big(I + [\gamma_s^+ - \gamma_s^-]N_s\Big)\,\partial_t\eta_2 + \alpha\Delta_s^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 = -\gamma_s^+ N_v\mathbf{v} + \gamma_s^- N_v\mathbf{v}. \tag{3.3.24}$$

Then, thanks to the identity $\mathbf{v} = P\mathbf{v} + (I-P)\mathbf{v}$ and the relation $(I-P)\mathbf{v} = (I-P)D_s^v\zeta_2$, we deduce that equation (3.3.24) can be rewritten as follows:

$$\begin{aligned}\Big(I + [\gamma_s^+ - \gamma_s^-]N_s\Big)\,\partial_t\eta_2 + \alpha\Delta_s^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 = &- \gamma_s^+ N_v(P\mathbf{v} + D_s^v\eta_2 - PD_s^v\eta_2)\\ &+ \gamma_s^- N_v(P\mathbf{v} + D_s^v\eta_2 - PD_s^v\eta_2).\end{aligned} \tag{3.3.25}$$

Lemma 3.3.2. *Let $\alpha \in (0,\alpha^*)$. The operator $K_s = I + (\gamma_s^+ - \gamma_s^-)N_s$ is an automorphism in $L^2(0,\ell_s)$. Moreover, K_s is an automorphism in $H^\alpha(0,\ell_s)$.*

Proof. We set $L_s = (\gamma_s^+ - \gamma_s^-)N_s$. We observe that since $\gamma_s^+ N_s$ and $\gamma_s^- N_s$ belong to $\mathcal{L}(L^2(0,\ell_s), H^{\frac{1}{2}}(0,\ell_s))$, the operator L_s is a compact operator in $L^2(0,\ell_s)$. Thus, to conclude is enough to show that L_s is positive. Let us consider $q = N_s\kappa$. After integrating by parts we get

$$0 = \int_\Omega \Delta q\, q = -\int_\Omega |\nabla q|^2 + \int_{\partial\Omega} q\frac{\partial q}{\partial n} = -\int_\Omega |\nabla q|^2 + \int_0^{\ell_s} [(\gamma_s^+ - \gamma_s^-)N_s\kappa]\kappa,$$

from where we deduce that the operator L_s is non-negative. Moreover, if $L_s\kappa = 0$, then $q = 0$ in Ω, because $q = 0$ on Γ_n. Therefore, L_s is positive. On the other hand, given $q \in H^\alpha(0,\ell_s)$, we know that there exists a unique $\zeta \in L^2(0,\ell_s)$ such that $q = K_s\zeta = \zeta + (\gamma_s^+ - \gamma_s^-)N_s\zeta$. Since $N_s\zeta \in H^1(\Omega)$, we have that $(\gamma_s^+ - \gamma_s^-)N_s\zeta \in H^{\frac{1}{2}}(0,\ell_s)$. Finally, since $\alpha \in (0,\alpha^*)$, we deduce that $\zeta \in H^\alpha(0,\ell_s)$. This completes the proof. □

Now, we will proceed to write the system satisfied by the triple $(P\mathbf{v},\zeta_1,\zeta_2)$. We consider the Hilbert space

$$\mathbf{Z} = \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega) \times H_s, \tag{3.3.26}$$

equipped with the inner product

$$\langle(\mathbf{u},\eta_1,\eta_2),(\mathbf{v},\zeta_1,\zeta_2)\rangle_{\mathbf{Z}} = \langle\mathbf{u},\mathbf{v}\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)} + +\langle(\eta_1,\eta_2),(\zeta_1,\zeta_2)\rangle_{H_s},$$

where $\langle\cdot,\cdot\rangle_{H_s}$ is defined in (3.3.18). We recall that the cut-off function Ψ is introduced in (3.2.4).

We set $\mathbf{V}_{n,\Gamma_d}^{\frac{1}{2}+\alpha}(\Omega) = \mathbf{H}_{\Gamma_d}^{\frac{1}{2}+\alpha}(\Omega) \cap \mathbf{V}_{n,\Gamma_d}^0(\Omega)$. We now define the unbounded operator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ in $\mathbf{Z}$ by

$$\mathcal{D}(\mathcal{A}) = \Big\{(P\mathbf{v},\zeta_1,\zeta_2) \in \mathbf{V}_{n,\Gamma_d}^{\frac{1}{2}+\alpha}(\Omega) \times \mathcal{D}(A_s) \mid P\mathbf{v} - PD_s^v\zeta_2 \in \mathcal{D}(A_0;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))\Big\},$$

and

$$\mathcal{A} = A_1 + B_1^+ + B_1^- + B_2 + B_3, \tag{3.3.27}$$

where

$$A_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s^v \\ 0 & 0 & I \\ 0 & -\alpha\Delta_s^2 & -\gamma(\Delta_s^2)^{1/2}\end{pmatrix}, \quad \mathcal{D}(A_1) = \mathcal{D}(\mathcal{A}), \tag{3.3.28}$$

$$B_1^\pm\begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K_s^{-1}\gamma_s^\pm N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2)\end{pmatrix}, \quad \mathcal{D}(B_1^\pm) = \mathcal{D}(\mathcal{A}), \tag{3.3.29}$$

$$B_2 \begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(K_s^{-1} - I)\alpha\Delta_s^2\zeta_1 \end{pmatrix}, \quad \mathcal{D}(B_2) = \mathcal{D}(\mathcal{A}), \tag{3.3.30}$$

$$B_3 \begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(K_s^{-1} - I)\gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 \end{pmatrix}, \quad \mathcal{D}(B_3) = \mathcal{D}(\mathcal{A}). \tag{3.3.31}$$

Remark 10. Concerning the operator $(B_1^{\pm}, \mathcal{D}(B_1^{\pm}))$, we remark that, at least formally,

$$B_1^{+} \begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K_s^{-1}\gamma_s^{+} N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K_s^{-1}\gamma_s^{+}[\Psi N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2)] \end{pmatrix},$$

where Ψ is the cut-off function introduced in (3.2.4). The precise definition of the trace of $\Psi N_v(P\mathbf{v} + PD_s\eta_2 - D_s\eta_2)$ is established later in (3.3.38).

The following result is a consequence of Proposition 3.3.4:

Theorem 3.3.6. *Let $\alpha \in (0, \alpha^*)$, $\delta \in (\delta^*, 1)$ and $0 < T < 1$. Assume that $\zeta_1 \in H^{4,2}((0,T) \times (0,\ell_s))$, $\zeta_2 \in H^{2,1}((0,T) \times (0,\ell_s))$, $\mathbf{v} \in L^2(0,T; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,T; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$ and $p \in L^2(0,T; H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$. Suppose that $\mathbf{F}_f = 0$, $\mathbf{G}_{\text{div}} = \mathbf{0}$, $\mathbf{g}^i = \mathbf{0}$ and $F_s = 0$ in (3.3.1). Then $(\mathbf{v}, p, \zeta_1, \zeta_2)$ is a solution of (3.3.1), if and only if,*

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} P\mathbf{v} \\ \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad \begin{pmatrix} P\mathbf{v}(0) \\ \zeta_1(0) \\ \zeta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}_0 \\ 0 \\ \zeta_2^0 \end{pmatrix}, \\ (I-P)\mathbf{v} = (I-P)D_s^v\zeta_2, \\ p = N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2) + N_s\partial_t\zeta_2. \end{cases} \tag{3.3.32}$$

3.3.4 Analyticity of the semigroup generated by $(\mathcal{A}, \mathcal{D}(A))$ on $\mathbf{Z}$

The main result of this section is the following theorem:

Theorem 3.3.7. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$.*

In order to show the theorem, we will first to prove that the operator $(A_1, \mathcal{D}(A_1))$, with $\mathcal{D}(A_1) = \mathcal{D}(\mathcal{A})$, is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$. Then, thanks to a perturbation result of generators of analytic semigroups is enough to show that the operators $(B_1^+, \mathcal{D}(A)), (B_1^-, \mathcal{D}(A)), (B_2, \mathcal{D}(A))$ and $(B_3, \mathcal{D}(A))$ are A_1-bounded with relative bound zero (cf. A. Pazy [Paz83, Chapter 3, Theorem 2.1] or K-J. Engel and R. Nagel [EN06, Chapter III, Theorem 2]).

For $a \in \mathbb{R}$ and $\theta \in (0, \pi)$, we define the sector $\Sigma_{a,\theta}$ by

$$\Sigma_{a,\theta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| < \theta\}.$$

Theorem 3.3.8. *The following assertions hold:*

(*i*) *There exists $a \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that the sector $\Sigma_{a,\theta}$ is contained in the resolvent set $\rho(A_1)$ of the unbounded operator $(A_1, \mathcal{D}(A_1))$. Moreover, there exists a positive constant C such that*

$$\|(\lambda I - A_1)^{-1}\|_{\mathcal{L}(\mathbf{Z})} \leq \frac{C}{|\lambda - a|}, \quad \textit{for all } \lambda \in \Sigma_{a,\theta} \setminus \{0\}. \tag{3.3.33}$$

(*ii*) *The domain $\mathcal{D}(\mathcal{A})$ of the fluid-structure operator defined in (3.3.27) is dense in* $\mathbf{Z}$.

(*iii*) *The unbounded operator $(A_1, \mathcal{D}(A_1))$ is the infinitesimal generator of an analytic semigroup on* $\mathbf{Z}$.

Proof.

(*i*) The proof is divided into two parts.

• **Study of the resolvent set $\rho(A_1)$.** Thanks to Theorem 3.3.3, we know that there exists $\theta_f \in (\pi/2, \pi)$ such that the sector Σ_{0,θ_f} is contained in $\rho(A_0)$. On the other hand, [CT89, Proposition 3.1] implies the existence of $a_s \in \mathbb{R}$ and $\theta_s \in (\pi/2, \pi)$ such that the sector Σ_{a_s,θ_s} is contained in $\rho(A_s)$.

We now set the sector $\Sigma_{a,\theta}$, where $a = \max\{0, a_s\}$ and $\theta = \min\{\theta_f, \theta_s\}$. Given $\lambda \in \Sigma_{a,\theta}$ and $(\mathbf{F}, g, h) \in \mathbf{Z}$, we consider the operator equation $(\lambda I - A_1)(P\mathbf{v}, \eta_1, \eta_2)^\top = (\mathbf{F}, g, h)^\top$. Following the proof given in [Ray10, Theorem 3.6, p. 5411], we have that

$$\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (\lambda I - A_0)^{-1} & 0 & ((\lambda I - A_0)^{-1}(-A_0)PD_s^v)\,(\lambda I - A_s)^{-1} \\ 0 & & (\lambda I - A_s)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{F} \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathcal{A}).$$

• **Estimate of the resolvent of $(A_1, \mathcal{D}(\mathcal{A}))$.** The proof is similar to the one presented in [Ray10, Theorem 3.6, p. 5411].

(*ii*) The proof makes use of Proposition 2.4.9 presented in the Appendix B at the end of Chapter 2. We consider two cases.

• Suppose that $a = 0$. Then, since $(0, \infty) \subset \Sigma_{a,\theta}$, from estimate (3.3.33) we deduce that

$$\|(\lambda I - A_1)\mathbf{v}\|_{\mathbf{Z}} \geq \lambda \widetilde{C} \|\mathbf{v}\|_{\mathbf{Z}} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A) \text{ and for all } \lambda > 0. \tag{3.3.34}$$

On the other hand, the surjectivity of $I - A_1$ follows from the first part of the proof of assertion (*i*). Thus, from Proposition 2.4.9 we conclude that $\mathcal{D}(A)$ is dense in $\mathbf{Z}$.

• Suppose that $a = a_s$. In this case, from (3.3.33) we deduce that

$$\|(\widetilde{\lambda} I - \widetilde{A}_1)\mathbf{v}\|_{\mathbf{Z}} \geq \widetilde{\lambda} \widetilde{C} \|\mathbf{v}\|_{\mathbf{Z}} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A) \text{ and for all } \widetilde{\lambda} > 0, \tag{3.3.35}$$

where $\widetilde{A}_1 := A_1 - aI$, with $\mathcal{D}(A_1) = \mathcal{D}(A)$. Once again, the surjectivity of the operator $I - \widetilde{A}_1$ follows from the first part of the proof of assertion (*i*). Thus, the densitiy of $\mathcal{D}(A)$ in $\mathbf{Z}$ follows from Proposition 2.4.9.

(*iii*) According to [EN06, Theorem 4.6, p. 95], it suffices to show that the operator $(A_1, \mathcal{D}(A))$ is sectorial and densely defined. These two properties are established in the assertions (*i*) and (*ii*), respectively.

□

Theorem 3.3.9. *The operators $(B_1^+, \mathcal{D}(\mathcal{A}))$ and $(B_1^-, \mathcal{D}(\mathcal{A}))$ are A_1-bounded with relative bound zero, i.e., for all $a > 0$, there exists $C_a > 0$ such that*

$$\|B_1^\pm (P\mathbf{v}, \eta_1, \eta_2)^\top\|_{\mathbf{Z}} \leq a \|A_1 (P\mathbf{v}, \eta_1, \eta_2)^\top\|_{\mathbf{Z}} + C_a \|(P\mathbf{v}, \eta_1, \eta_2)^\top\|_{\mathbf{Z}},$$

for all $(P\mathbf{v}, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A})$.

Proof. We will only consider the operator $(B_1^+, \mathcal{D}(\mathcal{A}))$. Thanks to [EN06, Lemma 2.13, p. 132] and [EN06, 2.15(i), p. 134], it is suffices to show that $B_1^+ \in \mathcal{L}(\mathcal{D}(A), \mathbf{Z})$ is a compact operator.

Let us first notice that

$$B_1^+\begin{pmatrix}P\mathbf{v}\\ \zeta_1\\ \zeta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\\ K_s^{-1}\gamma_s^+N_v(P\mathbf{v}-PD_s^v\zeta_2+D_s^v\zeta_2)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ K_s^{-1}\gamma_s^+[\Psi N_v(P\mathbf{v}-PD_s^v\zeta_2+D_s^v\zeta_2)]\end{pmatrix}, \tag{3.3.36}$$

where Ψ is the cut-off function introduced in (3.2.4). From Theorem 3.3.8 (i), we know that for all $\lambda\in\mathbb{R}$ satisfiying $\lambda>a$, where a is the parameter appearing in the cited theorem, and for all $(\mathbf{F},g,h)^\top\in\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, there exists $(P\mathbf{v},\eta_1,\eta_2)^\top\in\mathcal{D}(\mathcal{A})$ such that

$$(\lambda I-A_1)\begin{pmatrix}P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\mathbf{F}\\ g\\ h\end{pmatrix}. \tag{3.3.37}$$

Assume that $(I-P)\mathbf{v}=(I-P)D_s^v\eta_2$. We will show that there exists $C>0$ such that

$$\|\Psi N_v(P\mathbf{v}-PD_s^v\zeta_2+D_s^v\zeta_2)\|_{H^{\frac{1}{2}+\alpha}(\Omega)}\le C\|\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)}. \tag{3.3.38}$$

We proceed by density. We assume that $\mathbf{F}\in\mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Then, $(\mathbf{v},p,\eta_1,\eta_2)$ satisfies

$$\begin{cases}\lambda\mathbf{v}-\operatorname{div}\sigma(\mathbf{v},p)=\mathbf{F},\ \ \operatorname{div}\mathbf{v}=0\ \text{ in }\Omega,\\ \mathbf{v}=\eta_2\vec{\mathbf{e}}_2\ \text{ on }\Gamma_s,\ \ \mathbf{v}=0\ \text{ on }\Gamma_d\setminus\Gamma_s,\ \ \sigma(\mathbf{v},p)\mathbf{n}=0\ \text{ on }\Gamma_n,\\ \lambda\eta_1-\eta_2=g\ \text{ in }(0,\ell_s),\\ \lambda\eta_2+\alpha\Delta_s^2\eta_1+\gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2=h\ \text{ in }(0,\ell_s),\\ \eta_1(0)=\eta_{1,x}(0)=0\ \text{ and }\ \eta_{1,xx}(\ell_s)=\eta_{1,xxx}(\ell_s)=0.\end{cases} \tag{3.3.39}$$

We solve the above system by first solving the structure equation. It is possible to check that $(\eta_1,\eta_2)\in H^4_{\{0,\ell_s\}}(0,\ell_s)\times H^2_{\{0\}}(0,\ell_s)$. Moreover, there exists $C_1>0$ such that

$$\|\eta_1\|_{H^4_{\{0,\ell_s\}}(0,\ell_s)}+\|\eta_2\|_{H^2_{\{0\}}(0,\ell_s)}\le C_1\left(\|g\|_{H^2_{\{0\}}(0,\ell_s)}+\|h\|_{L^2(0,\ell_s)}\right).$$

In particular, we can choose $C_2>0$ large enough such that

$$\|\eta_2\|_{H^2_{\{0\}}(0,\ell_s)}\le C_1\left(\|g\|_{H^2_{\{0\}}(0,\ell_s)}+\|h\|_{L^2(0,\ell_s)}\right)\le C_2\|\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)}. \tag{3.3.40}$$

Then, using Theorem 3.3.1, we obtain that $\mathbf{v}\in\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and $p\in H^{\frac{1}{2}+\alpha}(\Omega)$. In addition, by using the same argument as in Proposition 3.3.4, we can express the pressure p as

$$p=N_v(P\mathbf{v}-PD_s^v\zeta_2+D_s^v\zeta_2)+\lambda N_s\eta_2. \tag{3.3.41}$$

We now set $\mathbf{v}_1=\Psi\mathbf{v}$ and $p_1=\Psi p$. We notice that the couple $(\mathbf{v}_1,p_1)$ solves

$$\begin{cases}\lambda\mathbf{v}_1-\operatorname{div}\sigma(\mathbf{v}_1,p)=\mathbf{F}_1,\ \ \operatorname{div}\mathbf{v}_1=\kappa\ \text{ in }\Omega,\\ \mathbf{v}_1=\Psi\eta_2\vec{\mathbf{e}}_2\ \text{ on }\Gamma_s,\ \ \mathbf{v}=0\ \text{ on }(\Gamma_d\setminus\Gamma_s)\cup\Gamma_n,\end{cases} \tag{3.3.42}$$

where

$$\mathbf{F}_1=\Psi\mathbf{F}+p\nabla\Psi-\nu\left(\mathbf{v}\Delta\Psi+2(\nabla\Psi\cdot\nabla)\mathbf{v}+\nabla(\mathbf{v}\cdot\nabla\Psi)\right)\ \text{ and }\ \kappa=\nabla\Psi\cdot\mathbf{v}.$$

Since $\Psi\mathbf{F} \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, $p \in L^2(\Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$, we deduce that

$$\mathbf{F}_1 \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega), \quad \Psi\eta_2\vec{\mathbf{e}}_2 \in \mathbf{H}^{\frac{3}{2}}(\Gamma_d) \ \text{ and } \ \kappa \in H^{\frac{1}{2}+\alpha}(\Omega).$$

Next, from [Dau89, Theorem 5.5(a)] and [BR, Theorem 3.2 and Corollary 3.3], we deduce in particular that there exists $C_3 > 0$ such that

$$\|p_1\|_{H^{\frac{1}{2}+\alpha}(\Omega)} \leq C_3\|\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)} \tag{3.3.43}$$

Thus, using (3.3.40) and (3.3.43) to estimate the terms appearing in the equality (3.3.41), we get

$$\|\Psi N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2)\|_{H^{\frac{1}{2}+\alpha}(\Omega)} \leq \|\lambda\Psi N_s\eta_2\|_{H^{\frac{1}{2}+\alpha}(\Omega)} + \|\Psi p\|_{H^{\frac{1}{2}+\alpha}} \leq C\|\mathbf{F}\|_{\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)}.$$

The last estimate, together with the density of $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$ in $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, allows us to conclude estimate (3.3.38). Now, since $\gamma_s^+[\Psi N_v(P\mathbf{v} - PD_s^v\zeta_2 + D_s^v\zeta_2)] \in H^\alpha(0,\ell_s)$, from the fact that $K_s^{-1} \in \mathcal{L}(H^\alpha(0,\ell_s))$ and that the embedding $H^\alpha(0,\ell_s) \hookrightarrow L^2(0,\ell_s)$ is compact, we deduce that $B_1^+ \in \mathcal{L}(\mathcal{D}(A),\mathbf{Z})$ is a compact operator. This completes the proof. □

The proof of the following theorem may be adapted from [Ray10, Lemma 3.9].

Theorem 3.3.10. *There exists $\theta_1, \theta_2 \in (0,1)$ such that $B_2 \in \mathcal{L}(\mathcal{D}((-\mathcal{A})^{\theta_1}),\mathbf{Z})$ and $B_3 \in \mathcal{L}(\mathcal{D}((-\mathcal{A})^{\theta_2}),\mathbf{Z})$.*

Proof of Theorem 3.3.7. From assertion (iii) of Theorem 3.3.8 we know that the operator $(A_1,\mathcal{D}(A))$ generates an analytic semigroup on $\mathbf{Z}$. Then, using Theorems 3.3.9, 3.3.10 and [Paz83, Chapter 3, Theorem 2.1] (see also [EN06, Chapter III, Theorem 2]), we deduce that the fluid-structure operator $(A,\mathcal{D}(A))$ generates an analytic semigroup on $\mathbf{Z}$. This completes the proof of Theorem 3.3.7.

3.4 Existence and regularity results for the non-homogeneous linear system

Before presenting the result of the well-posedness for system (3.3.1), we will start by introducing an appropriate lifting associated to the inflow data $\mathbf{g}^i$ on Γ_i, and another one associated to the data $\mathbf{G}_{\text{div}}$ that appears in the divergence condition in system (3.3.1).

- Lifting of the inflow data $\mathbf{g}^i$ on Γ_i. Let us consider the following system:

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{z}(t),\pi(t)) = 0 \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{z}(t) = 0 \ \text{ in } \Omega, \\ \mathbf{z}(t) = \mathbf{g}^i(t) \ \text{ on } \Gamma_i, \quad \mathbf{z}(t) = 0 \ \text{ on } \Gamma_r\cup\Gamma_s\cup\Gamma_w, \\ \sigma(\mathbf{z}(t),\pi(t))\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{3.4.1}$$

The following proposition is a direct consequence of Theorem 3.3.1 (ii).

Proposition 3.4.1. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*,1)$ and $0 < T < 1$. For all $\mathbf{g}^i \in H^1_{\{0\}}(0,T;\mathbf{H}(\Gamma_i))$, system (3.4.1) admits unique solution $(\mathbf{z},\pi) \in H^1(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \times H^1(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$. Moreover, there exists a positive constant $C_{\alpha,\delta}$, independent of T, such that*

$$\|\mathbf{z}\|_{H^1(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))} + \|\pi\|_{H^1(0,T;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))} \leq C_{\alpha,\delta}\|\mathbf{g}^i\|_{H^1_{\{0\}}(0,T;\mathbf{H}(\Gamma_i))}. \tag{3.4.2}$$

• Lifting of the divergence data $\mathbf{G}_{\text{div}}$ in Ω. We seek a lifting $\mathbf{w}$ of the data $\mathbf{G}_{\text{div}}$ appearing in the divergence condition in (3.3.1) satisfying

$$\mathbf{w} \in L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)). \tag{3.4.3}$$

First, we explain why the approaches used in [FNR19, Theorem 10.2] and [MRR20, Proposition 5.1] cannot be directly applied separetely to our setting. Then, we show how these two strategies can be combined to define the desired lifting.

First, unlike the lifting introduced in [FNR19, Theorem 10.2], where direct conditions on $\mathbf{G}_{\text{div}}$ are imposed, it is not possible to apply this idea to our setting. Indeed, if we follow this approach, we should assume that

$$\mathbf{G}_{\text{div}} \in L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha}(\Omega)) \cap H^1(0,T;\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega)).$$

However, in the fixed point argument we cannot estimate all of the terms involved in $\mathbf{G}_{\text{div}}$ in those norms. For instance, the estimation of the term $(\partial_{z_1}\eta_1^\pm)u_1$ in the $H^{\frac{3}{2}+\alpha}(\Omega)-$norm would require that $\partial_{z_1}\eta_1^\pm \in H^{\frac{5}{2}+\alpha}(-L/2,L)$, which is not possible due to Proposition 3.2.1. For the definition of $\mathbf{G}_{\text{div}}$, we refer the reader to (3.6.22).

A second approach is the one proposed in [MRR20, Proposition 5.1], where the authors construct a lifting based on a quasi-stationary Stokes system. More precisely, the lifting $\mathbf{w}$ is solution of the equation

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{w},\pi) = \mathbf{F} \ \text{ in } \Omega, \ \ \operatorname{div}\mathbf{w} = \operatorname{div}\mathbf{G}_{\text{div}} \ \text{ in } \Omega, \\ \mathbf{w} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_n. \end{cases} \tag{3.4.4}$$

The way to adapt that idea to our setting is by requiring the following regularity:

$$\operatorname{div}\mathbf{G}_{\text{div}} \in L^2(0,T;\mathbf{H}^{\frac{1}{2}+\alpha,1}(\Omega)) \ \text{ and } \ \mathbf{G}_{\text{div}} \in H^1(0,T;\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega)).$$

Nevertheless, the problem with this approach arises from the fact that we cannot ensure the regularity $\mathbf{w} \in H^1(0,T;\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega))$ from the condition $\mathbf{G}_{\text{div}} \in H^1(0,T;\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega))$. Indeed, since $\mathbf{G}_{\text{div}} \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, we must define the solution $\mathbf{w}$ of (3.4.4) by transposition. In doing so, we obtain that $\mathbf{w}$ and the adjoint states $(\mathbf{\Phi},\psi)$ satisfy

$$\langle \mathbf{w},\boldsymbol{\zeta}\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha},\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}} = \langle \mathbf{G}_{\text{div}},\nabla\psi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha},\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}}, \tag{3.4.5}$$

for all $\boldsymbol{\zeta} \in \mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)$, where $(\mathbf{\Phi},\psi)$ is solution of system

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{\Phi},\psi) = \boldsymbol{\zeta} \ \text{ in } \Omega, \ \ \operatorname{div}\mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{\Phi},\psi)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{3.4.6}$$

We note that the right-hand side of identity (3.4.5) is well-defined provided $\psi \in H^{\frac{3}{2}-\alpha}(\Omega)$. However, we cannot ensure this regularity because of the reentrant corners present in the domain fluid Ω.

Let us now explain how we combine the approaches mentioned above in order to obtain a lifting $\mathbf{w}$ with the desired regularity (3.4.3). Roughly speaking, the idea consists in splitting the fluid domain into two parts, $\mathcal{O}_L$ and $\mathcal{O}_R$, as shown in Figure 3.5. Then, in the region $\mathcal{O}_L$, which does not include the reentrant corners, we apply the strategy of [MRR20, Proposition 5.1], based on a quasi-stationary Stokes system, while in the region $\mathcal{O}_R$ we use the approach described in [FNR19, Theorem 10.2].

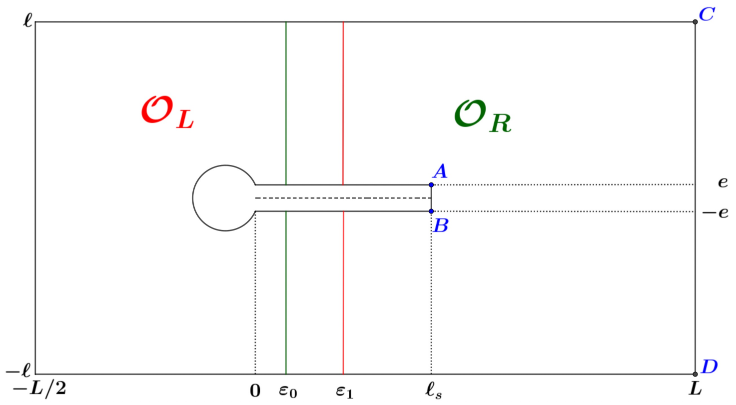


Figure 3.5 – Representation of the split fluid domain Ω.

Let us start by introducing some notation. Let ε_0 and ε_1 be two positive constants satisfying $\varepsilon_0 < \varepsilon_1 < \ell_s$. Let us set

$$\begin{aligned}
\mathcal{O}_R &= ([\varepsilon_0, \ell_s] \times ([-\ell, -e] \cup [e, \ell])) \cup ([\ell_s, L] \times [-\ell, \ell]) \,, \\
\mathcal{O}_L &= (\Omega \setminus \mathcal{O}_R) \cup ([\varepsilon_0, \varepsilon_1] \times ([-\ell, -e] \cup [e, \ell])) \,.
\end{aligned}$$

We introduce the cut-off function $\widetilde{\theta} \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $0 \leq \widetilde{\theta} \leq 1$,

$$\widetilde{\theta} = 1 \text{ on } [-L/2, \varepsilon_0] \times [-\ell, \ell] \;\text{ and }\; \widetilde{\theta} = 0 \text{ on } [\varepsilon_1, L] \times [-\ell, \ell]. \tag{3.4.7}$$

We consider the following decomposition of $\mathbf{G}_{\text{div}}$:

$$\mathbf{G}_{\text{div}} = \widetilde{\theta}\mathbf{G}_{\text{div}} + (1 - \widetilde{\theta})\mathbf{G}_{\text{div}}. \tag{3.4.8}$$

Let $(\mathbf{w}_L(t), p_L(t))$ be the solution of (3.3.2) with $\mathbf{F} = 0$, $\mathbf{g} = 0$ and $h = \text{div}(\widetilde{\theta}\mathbf{G}_{\text{div}})$, where $\widetilde{\theta}\mathbf{G}_{\text{div}}$ satisfy

$$\text{div}(\widetilde{\theta}\mathbf{G}_{\text{div}}) \in L^2(0, T; H^{\frac{1}{2}+\alpha,1}(\Omega)) \tag{3.4.9}$$

and

$$\widetilde{\theta}\mathbf{G}_{\text{div}} \in H^1(0, T; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)). \tag{3.4.10}$$

Let $\mathbf{w}_R := (1 - \widetilde{\theta})\mathbf{G}_{\text{div}}$, with $(1 - \widetilde{\theta})\mathbf{G}_{\text{div}}$ satisfying

$$\begin{aligned}
&(1 - \widetilde{\theta})\mathbf{G}_{\text{div}} \in L^2(0, T; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0, T; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)), \\
&(1 - \widetilde{\theta})\mathbf{G}_{\text{div}} = 0 \;\text{ in } \Omega_2, \\
&(1 - \widetilde{\theta})\mathbf{G}_{\text{div}} = 0 \;\text{ on } \Sigma_d^T, \;\; \varepsilon((1 - \widetilde{\theta})\mathbf{G}_{\text{div}})\mathbf{n} = 0 \;\text{ on } \Sigma_n^T.
\end{aligned} \tag{3.4.11}$$

Let us now set

$$\mathbf{w} := \mathbf{w}_L + \mathbf{w}_R. \tag{3.4.12}$$

As a consequence of Theorem 3.3.1 and Lemma 3.C.2, we have the following result:

Proposition 3.4.2. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*,1)$ and $0 < T < 1$. Under the assumptions (3.4.9) and (3.4.10), along with condition (3.4.11), the function $\mathbf{w}$ defined in (3.4.12) belongs to $L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$. Moreover, there exists a constant $C_{\alpha,\delta} > 0$ independent of T, such that*

$$
\begin{aligned}
&\|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega))} + \|\mathbf{w}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&\le C_{\alpha,\delta}\Big(\Big\|\mathrm{div}\,\big(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\big)\Big\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&\qquad\qquad + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\Big).
\end{aligned}
\tag{3.4.13}
$$

We are now in position to establish the main result of this section.

Theorem 3.4.1. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*,1)$ and $0 < T < 1$. Let us assume that $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)$, $\zeta_2^0 \in H^1_{\{0\}}(0,\ell_s)$, $\mathbf{F}_f \in L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$, $\mathbf{g}^i \in H^1_{\{0\}}(0,1;\mathbf{H}(\Gamma_i))$ and $F_s \in L^2(0,T;L^2(0,\ell_s))$. Let us assume that $\mathbf{G}_{\mathrm{div}}$ satisfies assumptions (3.4.9) and (3.4.10), along with condition (3.4.11) and $\mathbf{G}_{\mathrm{div}}|_{t=0} = 0$, and let $\mathbf{w}$ denotes its lifting defined as in (3.4.12). We assume in addition the following conditions:*

$$
\begin{cases}
\mathbf{v}_0 = 0 \ \text{ on } \Gamma_i, \quad \mathbf{v}_0 = 0 \ \text{ on } \Gamma_r \cup \Gamma_w, \\
\mathbf{v}_0 = \zeta_2(0,\cdot)\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathrm{div}\,\mathbf{v}_0 = 0 \ \text{ in } \Omega, \\
(P\mathbf{v}_0, 0, \zeta_2^0) \in [\mathcal{D}(\mathcal{A}), \mathbf{Z}]_{1/2}, \quad P\mathbf{v}_0 \in [\mathcal{D}(A_0; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathbf{V}^0_{n,\Gamma_d}(\Omega)]_{1/2},
\end{cases}
\tag{3.4.14}
$$

Then, system (3.3.1) admits a unique solution $(\mathbf{v}, p, \zeta_1, \zeta_2)$ belonging to $L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \times L^2(0,T;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)) \times H^{4,2}((0,T)\times(0,\ell_s)) \times H^{2,1}((0,T)\times(0,\ell_s))$, satisfying the estimate

$$
\begin{aligned}
&\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega))} + \|\mathbf{v}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|p\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega))} \\
&+ \|\zeta_1\|_{H^{4,2}((0,T)\times(0,\ell_s))} + \|\zeta_2\|_{H^{2,1}((0,T)\times(0,\ell_s))} \\
&\le C_L\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{g}^i\|_{L^2(0,1;\mathbf{H}(\Gamma_i))} + \|\partial_t\mathbf{g}^i\|_{L^2(0,1;\mathbf{H}(\Gamma_i))} + \|\zeta_2^0\|_{H^1(0,\ell_s)} \\
&\qquad + \|\mathbf{F}_\mathbf{f}\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|F_s\|_{L^2((0,T)\times(0,\ell_s))} \\
&\qquad + \Big\|\mathrm{div}\,\big(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\big)\Big\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&\qquad + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\Big),
\end{aligned}
\tag{3.4.15}
$$

where $C_L > 0$ is a constant independent of T.

Proof. We split the proof in two steps.

• **Step 1**. *Reformulation of the system (3.3.1).*

Let $(\mathbf{z}(t), \pi(t))$ be the solution to system (3.4.1). Let $\mathbf{w}$ be the vector field given in (3.4.12). We set $\widetilde{\mathbf{v}} = \mathbf{v} - \mathbf{w} - \mathbf{z}$ and $\widetilde{p} = p - p_L - \pi$. Next, the couple $(\widetilde{\mathbf{v}}, \widetilde{p})$ satisfies

$$
\begin{cases}
\partial_t \widetilde{\mathbf{v}} - \operatorname{div} \sigma(\widetilde{\mathbf{v}}, \widetilde{p}) = \mathbf{F}_f - \partial_t \mathbf{w} + 2\nu \operatorname{div} \varepsilon(\mathbf{w}) - \partial_t \mathbf{z} - \nabla p_L \ \text{ in } Q^T, \\
\operatorname{div} \widetilde{\mathbf{v}} = 0 \ \text{ in } Q^T, \\
\widetilde{\mathbf{v}} = 0 \ \text{ on } \Sigma_i^T, \ \ \widetilde{\mathbf{v}} = 0 \ \text{ on } \Sigma_w^T \cup \Sigma_r^T, \ \ \widetilde{\mathbf{v}} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \\
\sigma(\widetilde{\mathbf{v}}, \widetilde{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^T, \ \ \widetilde{\mathbf{v}}(0) = \mathbf{v}_0 - \mathbf{w}(0) - \mathbf{z}(0) \ \text{ in } \Omega, \\
\partial_t \zeta_1 = \zeta_2 \ \text{ in } (0,T) \times (0, \ell_s), \\
\partial_t \zeta_2 + \alpha \Delta_s^2 \zeta_1 + \gamma (\Delta_s^2)^{\frac{1}{2}} \zeta_2 = -\gamma_s^+ \widetilde{p} + \gamma_s^- \widetilde{p} + F_s \ \text{ in } (0,T) \times (0, \ell_s), \\
\zeta_1 = 0 \ \text{ and } \ \partial_{z_1} \zeta_1 = 0 \ \text{ on } (0,T) \times \{0\}, \\
\partial_{z_1}^2 \zeta_1 = 0 \ \text{ and } \ \partial_{z_1}^3 \zeta_1 = 0 \ \text{ on } (0,T) \times \{\ell_s\}, \\
\zeta_1(0) = 0 \ \text{ and } \ \zeta_2(0) = \zeta_2^0 \ \text{ in } (0, \ell_s).
\end{cases}
\tag{3.4.16}
$$

We notice that the structure equation can be rewritten as follows:

$$
\begin{aligned}
\Big(I + [\gamma_s^+ - \gamma_s^-] N_s\Big) \partial_t \zeta_2 + \alpha \Delta_s^2 \zeta_1 + \gamma (\Delta_s^2)^{\frac{1}{2}} \zeta_2 &= -\gamma_s^+ N_v (P \widetilde{\mathbf{v}} + D_s^v \zeta_2 - P D_s^v \zeta_2) \\
&\quad + \gamma_s^- N_v (P \widetilde{\mathbf{v}} + D_s^v \zeta_2 - P D_s^v \zeta_2) - \gamma_s^+ N_p \mathbf{F} + \gamma_s^- N_p \mathbf{F} \\
&\quad + \gamma_s^+ \pi - \gamma_s^- \pi + F_s,
\end{aligned}
$$

where $\mathbf{F} := \mathbf{F}_f - \partial_t \mathbf{w} + 2\nu \operatorname{div} \varepsilon(\mathbf{w}) - \partial_t \mathbf{z} - \nabla p_L$. Thus, from Theorem 3.3.6 we have that the solution $(\widetilde{\mathbf{v}}, \widetilde{p}, \zeta_1, \zeta_2)$ of (3.4.16) satisfies

$$
\begin{cases}
\dfrac{d}{dt} \begin{pmatrix} P\widetilde{\mathbf{v}} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} P\widetilde{\mathbf{v}} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{F} \\ 0 \\ H \end{pmatrix}, \quad \begin{pmatrix} P\widetilde{\mathbf{v}}(0) \\ \zeta_1(0) \\ \zeta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}_0 \\ 0 \\ \zeta_2^0 \end{pmatrix}, \\
(I - P)\widetilde{\mathbf{v}} = (I - P) D_s^v \zeta_2, \\
\widetilde{p} = N_v (P \widetilde{\mathbf{v}} - P D_s^v \zeta_2 + D_s^v \zeta_2) + N_s \zeta_2 + N_p \mathbf{F},
\end{cases}
\tag{3.4.17}
$$

where $H = K_s^{-1} \left(-\gamma_s^+ N_p \mathbf{F} + \gamma_s^- N_p \mathbf{F} + F_s + \gamma_s^+ \pi - \gamma_s^- \pi \right)$. We observe that

$$
\begin{aligned}
\|P\mathbf{F}\|_{L^2(0,T;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0})} &\leq C \Big(\|\mathbf{F}_f\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \|\partial_t \mathbf{w}\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \\
&\quad + \|\operatorname{div} \varepsilon(\mathbf{w})\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \|\mathbf{g}\|_{L^2(0,1;\mathbf{H}^{\frac{3}{2}}(\Gamma_i))} + \|\partial_t \mathbf{g}\|_{L^2(0,1;\mathbf{H}^{\frac{3}{2}}(\Gamma_i))} \Big)
\end{aligned}
\tag{3.4.18}
$$

and

$$
\begin{aligned}
\|H\|_{L^2(0,T;L^2(0,\ell_s))} &\leq C \Big(\|\mathbf{F}_f\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \|F_s\|_{L^2(0,T;L^2(0,\ell_s))} \\
&\quad + \|\mathbf{g}\|_{L^2(0,1;\mathbf{H}^{\frac{3}{2}}(\Gamma_i))} + \|\partial_t \mathbf{g}\|_{L^2(0,1;\mathbf{H}^{\frac{3}{2}}(\Gamma_i))} \Big),
\end{aligned}
\tag{3.4.19}
$$

where $C > 0$ is independent of T. In the last two estimates we used (3.4.2).

• **Step 2**. *Regularity of solutions to system* (3.4.16).

In order to show that the constants that appear in the estimates below are independent of T, we proceed as follows. Let us set

$$
\overline{\mathbf{F}} = \begin{cases} \mathbf{F} & \text{if } 0 < t \leq T, \\ 0 & \text{if } T < t \leq 1, \end{cases} \quad \text{and} \quad \overline{H} = \begin{cases} H & \text{if } 0 < t \leq T, \\ 0 & \text{if } T < t \leq 1. \end{cases}
$$

Notice that $P\overline{\mathbf{F}} \in L^2(0,1;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega))$ and $\overline{H} \in L^2((0,1) \times (0,\ell_s))$. Next, instead of system

(3.4.17), we consider the following problem:

$$\frac{d}{dt}\begin{pmatrix} P\overline{\mathbf{v}} \\ \overline{\zeta}_1 \\ \overline{\zeta}_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} P\overline{\mathbf{v}} \\ \overline{\zeta}_1 \\ \overline{\zeta}_2 \end{pmatrix} + \begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ \overline{H} \end{pmatrix}, \quad t\in(0,1), \qquad \begin{pmatrix} P\overline{\mathbf{v}}(0) \\ \overline{\zeta}_1(0) \\ \overline{\zeta}_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}_0 \\ 0 \\ \zeta_2^0 \end{pmatrix}.$$

Since $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$ (see Theorem 3.3.7) and $(P\mathbf{v}_0, 0, \zeta_2^0) \in [\mathcal{D}(\mathcal{A}), \mathbf{Z}]_{1/2}$, the maximal regularity result [BDDM07, Theorem 3.1, p. 143] implies that $(P\overline{\mathbf{v}}, \overline{\zeta}_1, \overline{\zeta}_2)$ belongs to $L^2(0,1;\mathcal{D}(\mathcal{A}))\cap H^1(0,1;\mathbf{Z})$ and satisfies

$$\begin{aligned} &\|P\overline{\mathbf{v}}\|_{H^1(0,1;\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d})} + \|\overline{\zeta}_1\|_{H^{4,2}((0,1)\times(0,\ell_s))} + \|\overline{\zeta}_2\|_{H^{2,1}((0,1)\times(0,\ell_s))} \\ &\le C\left(\|\mathbf{v}_0\|_{\mathbf{H}^1} + \|\zeta_2^0\|_{H^1(0,\ell_s)} + \|P\overline{\mathbf{F}}\|_{L^2(0,1;\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d})} + \|\overline{H}\|_{L^2(0,1;L^2(0,\ell_s))}\right), \end{aligned} \tag{3.4.20}$$

where C is independent of T. Let us notice that by construction we have $(P\widetilde{\mathbf{v}}, \zeta_1, \zeta_2) = (P\overline{\mathbf{v}}, \overline{\zeta}_1, \overline{\zeta}_2)$ in $[0,T]$. Then, from (3.4.18), (3.4.19) and (3.4.20) it follows that

$$\begin{aligned} &\|P\widetilde{\mathbf{v}}\|_{H^1(0,T;\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d})} + \|\zeta_1\|_{H^{4,2}((0,T)\times(0,\ell_s))} + \|\zeta_2\|_{H^{2,1}((0,T)\times(0,\ell_s))} \\ &\le \|P\overline{\mathbf{v}}\|_{H^1(0,1;\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d})} + \|\overline{\zeta}_1\|_{H^{4,2}((0,1)\times(0,\ell_s))} + \|\overline{\zeta}_2\|_{H^{2,1}((0,1)\times(0,\ell_s))} \\ &\le C\left(\|\mathbf{v}_0\|_{\mathbf{H}^1} + \|\zeta_2^0\|_{H^1(0,\ell_s)} + \|P\mathbf{F}\|_{L^2(0,T;\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d})} + \|H\|_{L^2(0,T;L^2(0,\ell_s))}\right). \end{aligned} \tag{3.4.21}$$

We analyze separetely the regularity of ζ_1, ζ_2 and $\widetilde{\mathbf{v}}, p$.

- Regularity of ζ_1 and ζ_2. The fact that $\zeta_1 \in H^{4,2}((0,T)\times(0,\ell_s))$ and $\zeta_2 \in H^{2,1}((0,T)\times(0,\ell_s))$ follows from (3.4.21).
- Regularity of $\widetilde{\mathbf{v}}$ and $\widetilde{p}$. From (3.4.21) we deduce that $P\widetilde{\mathbf{v}} \in H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$. We claim that $(I-P)\widetilde{\mathbf{v}} \in H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$. Indeed,

$$\begin{aligned} \|(I-P)\widetilde{\mathbf{v}}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} &= \|(I-P)D_s^v\zeta_2\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} \\ &\le C\|(I-P)D_s^v\zeta_2\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \\ &\le C\|\zeta_2\|_{H^{2,1}((0,T)\times(0,\ell_s))}. \end{aligned} \tag{3.4.22}$$

To deduce the last estimate we first use the transposition method to define the solution of the mixed Stokes system with Dirichlet data $\zeta_{2,t} \in L^2(0,T;L^2(0,\ell_s))$, and then we conclude by invoking the Riesz representation theorem. Thus, $\widetilde{\mathbf{v}} \in H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$.

Let us now consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{v}},\widetilde{p}) = \mathbf{F} - \partial_t\widetilde{\mathbf{v}}, \quad \operatorname{div}\widetilde{\mathbf{v}} = 0 \ \text{ in } \Omega, \\ \widetilde{\mathbf{v}} = \zeta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \widetilde{\mathbf{v}} = 0 \ \text{ on } \Gamma_i\cup\Gamma_r\cup\Gamma_w, \quad \sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases}$$

Since $\mathbf{F} - \partial_t\widetilde{\mathbf{v}} \in L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)) \subset L^2(0,T;\mathbf{H}^{-1}_{\Gamma_d}(\Omega))$ and $\zeta_2 \in L^2(0,T;H^2_{\{0\}}(0,\ell_s))$, Theorem 2.3.1 implies that

$$\widetilde{\mathbf{v}} \in L^2(0,T;\mathbf{H}^1(\Omega)) \ \text{ and } \ \widetilde{p} \in L^2(0,T;L^2(\Omega)).$$

Let us now consider

$$\begin{cases} \partial_t \widetilde{\mathbf{v}} - \operatorname{div} \sigma(\widetilde{\mathbf{v}}, \widetilde{p}) = \mathbf{F}, \ \ \operatorname{div} \widetilde{\mathbf{v}} = 0 \ \text{ in } Q^T, \\ \widetilde{\mathbf{v}} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \ \ \widetilde{\mathbf{v}} = 0 \ \text{ on } \Sigma_i^T \cup \Sigma_r^T \cup \Sigma_w^T, \ \ \sigma(\widetilde{\mathbf{v}}, \widetilde{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^T. \end{cases}$$

We will show that

$$\widetilde{\mathbf{v}} \in L^2(0,T; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,T; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \ \text{ and } \ \widetilde{p} \in L^2(0,T; H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)). \quad (3.4.23)$$

Let us set $(\widetilde{\mathbf{v}}_1, \widetilde{p}_1) := (\Psi \widetilde{\mathbf{v}}, \Psi \widetilde{p})$ and $(\widetilde{\mathbf{v}}_2, \widetilde{p}_2) := ((1-\Psi)\widetilde{\mathbf{v}}, (1-\Psi)\widetilde{p})$. To show (3.4.23), it is suffices to prove that

$$\widetilde{\mathbf{v}}_1 \in L^2(0,T; \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)) \cap H^1(0,T; \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)) \ \text{ and } \ \widetilde{p}_1 \in L^2(0,T; H^{\frac{1}{2}+\alpha}(\Omega)) \quad (3.4.24)$$

and

$$\widetilde{\mathbf{v}}_2 \in L^2(0,T; \mathbf{H}_\delta^2(\Omega)) \cap H^1(0,T; \mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{p}_2 \in L^2(0,T; H_\delta^1(\Omega)). \quad (3.4.25)$$

- Proof of (3.4.24). Firstly, since $\widetilde{\mathbf{v}} \in H^1(0,T; \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$, then $\widetilde{\mathbf{v}}_1 \in H^1(0,T; \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$. Let us now observe that $(\widetilde{\mathbf{v}}_1, \widetilde{p}_1)$ solves

$$\begin{cases} -\operatorname{div} \sigma(\widetilde{\mathbf{v}}_1, \widetilde{p}_1) = \mathbf{F}_1 - \partial_t \widetilde{\mathbf{v}}_1, \ \ \operatorname{div} \widetilde{\mathbf{v}}_1 = h_1 \ \text{ in } Q^T, \\ \widetilde{\mathbf{v}}_1 = \Psi \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \ \ \widetilde{\mathbf{v}}_1 = 0 \ \text{ on } \Sigma_i^T \cup \Sigma_r^T \cup \Sigma_w^T \cup \Sigma_n^T, \end{cases}$$

  where

$$\mathbf{F}_1 = \Psi \mathbf{F} + \widetilde{p} \nabla \Psi - \nu \left(\widetilde{\mathbf{v}} \Delta \Psi + 2(\nabla \Psi \cdot \nabla) \widetilde{\mathbf{v}} + \nabla(\widetilde{\mathbf{v}} \cdot \nabla \Psi) \right) \ \text{ and } \ h_1 = \nabla \Psi \cdot \widetilde{\mathbf{v}}.$$

  Since $\mathbf{F} \in L^2(0,T; \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$, $\widetilde{p} \in L^2(0,T; L^2(\Omega))$, $\widetilde{\mathbf{v}} \in L^2(0,T; \mathbf{H}^1(\Omega))$ and $\partial_t \widetilde{\mathbf{v}}_1 \in L^2(0,T; \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$, we deduce that

$$\begin{aligned} &\mathbf{F} - \partial_t \widetilde{\mathbf{v}}_1 \in L^2(0,T; \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)), \ \ h_1 \in L^2(0,T; H^{\frac{1}{2}+\alpha}(\Omega)) \\ &\text{and } \ \Psi \zeta_2 \mathbf{e}_2 \in L^2(0,T; \mathbf{H}^{\frac{3}{2}}(\Gamma_d)). \end{aligned}$$

  Then, after applying [Dau89, Theorem 5.5(a)] and [BR, Theorem 3.2 and Corollary 3.3], we obtain

$$\widetilde{\mathbf{v}}_1 \in L^2(0,T; \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)) \ \text{ and } \ \widetilde{p}_1 \in L^2(0,T; H^{\frac{1}{2}+\alpha}(\Omega)).$$

  This concludes the proof of (3.4.24).

- Proof of (3.4.25). Let us first observe that $(\widetilde{\mathbf{v}}_2, p_2)$ satisfies

$$\begin{cases} \partial_t \widetilde{\mathbf{v}}_2 - \operatorname{div} \sigma(\widetilde{\mathbf{v}}_2, \widetilde{p}_2) = \mathbf{F}_2, \ \ \operatorname{div} \widetilde{\mathbf{v}}_2 = h_2 \ \text{ in } Q^T, \\ \widetilde{\mathbf{v}}_2 = (1-\Psi) \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^T, \ \ \widetilde{\mathbf{v}}_2 = 0 \ \text{ on } \Sigma_d^T \setminus \Sigma_s^T, \ \ \sigma(\widetilde{\mathbf{v}}_2, p_2)\mathbf{n} = 0 \ \text{ on } \Sigma_n^T, \end{cases}$$

  where

$$\mathbf{F}_2 = (1-\Psi)\mathbf{F} - \widetilde{p} \nabla \Psi + \nu \left(\widetilde{\mathbf{v}} \Delta \Psi + 2(\nabla \Psi \cdot \nabla) \widetilde{\mathbf{v}} + \nabla(\widetilde{\mathbf{v}} \cdot \nabla \Psi) \right) \ \text{ and } \ h_2 = -\nabla \Psi \cdot \widetilde{\mathbf{v}}.$$

  Since $(1-\Psi)\mathbf{F} \in L^2(0,T; \mathbf{L}^2(\Omega))$, $\widetilde{p} \in L^2(0,T; L^2(\Omega))$, $\widetilde{\mathbf{v}} \in L^2(0,T; \mathbf{H}^1(\Omega))$, we deduce that

$$\mathbf{F}_2 \in L^2(0,T; \mathbf{L}^2(\Omega)) \ \text{ and } \ h_2 \in L^2(0,T; H^1(\Omega)).$$

We will now construct two lifting functions: one corresponding to the boundary data $(1-\Psi)\zeta_2\vec{\mathbf{e}}_2$, and another corresponding to the divergence data h_1.

• Lifting for the boundary data. Let us consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{w}}_1,\widetilde{\pi}_1)=0, \ \ \operatorname{div}\widetilde{\mathbf{w}}_1=0 \ \text{ in } \Omega, \\ \widetilde{\mathbf{w}}_1=(1-\Psi)\zeta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \widetilde{\mathbf{w}}_1=0 \ \text{ on } \Gamma_d\setminus\Gamma_s \ \ \sigma(\widetilde{\mathbf{w}}_1,\widetilde{\pi}_1)\mathbf{n}=0 \ \text{ on } \Gamma_n. \end{cases} \tag{3.4.26}$$

Since $\zeta_2\in H^{2,1}((0,T)\times(0,\ell_s))$, thanks to [MR10, Theorem 9.4.5] and the transposition method, we deduce that

$$\widetilde{\mathbf{w}}_1\in L^2(0,T;\mathbf{H}^2_\delta(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi}_1\in L^2(0,T;H^1_\delta(\Omega)). \tag{3.4.27}$$

• Lifting for the divergence data. Let us consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{w}}_2,\widetilde{\pi}_2)=0, \ \ \operatorname{div}\widetilde{\mathbf{w}}_2=h_2 \ \text{ in } \Omega, \\ \widetilde{\mathbf{w}}_2=0 \ \text{ on } \Gamma_d, \ \ \sigma(\widetilde{\mathbf{w}}_2,\widetilde{\pi}_2)\mathbf{n}=0 \ \text{ on } \Gamma_n. \end{cases} \tag{3.4.28}$$

Since $h_2\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-\frac12+\alpha}_{\Gamma_d}(\Omega))$, from [MR10, Theorem 9.4.5] and Lemma 3.6.1 we deduce that

$$\widetilde{\mathbf{w}}_2\in L^2(0,T;\mathbf{H}^2_\delta(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi}_2\in L^2(0,T;H^1_\delta(\Omega)). \tag{3.4.29}$$

Setting $\widetilde{\mathbf{w}}:=\widetilde{\mathbf{w}}_1+\widetilde{\mathbf{w}}_2$ and $\widetilde{\pi}:=\widetilde{\pi}_1+\widetilde{\pi}_2$, we deduce from (3.4.26) and (3.4.28) that

$$\widetilde{\mathbf{w}}\in L^2(0,T;\mathbf{H}^2_\delta(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi}\in L^2(0,T;H^1_\delta(\Omega)). \tag{3.4.30}$$

After setting $\widetilde{\mathbf{v}}_2=\widetilde{\mathbf{w}}+\widetilde{\mathbf{z}}$ and $\widetilde{p}_2=\widetilde{\pi}+\widetilde{q}$, we observe that $(\widetilde{\mathbf{z}},\widetilde{q})$ solves

$$\begin{cases} \partial_t\widetilde{\mathbf{z}}-\operatorname{div}\sigma(\widetilde{\mathbf{z}},\widetilde{q})=\mathbf{F}_2-\partial_t\widetilde{\mathbf{w}}, \ \ \operatorname{div}\widetilde{\mathbf{z}}=0 \ \text{ in } Q^T, \\ \widetilde{\mathbf{z}}=0 \ \text{ on } \Sigma^T_d, \ \ \sigma(\widetilde{\mathbf{z}},\widetilde{q})\mathbf{n}=0 \ \text{ on } \Sigma^T_n. \end{cases}$$

Since $\mathbf{F}_2-\partial_t\widetilde{\mathbf{w}}\in L^2(0,T;\mathbf{L}^2(\Omega))$ and the semigroup generated by the Stokes operator $(A_0;\mathbf{V}^0_{n,\Gamma_d}(\Omega))$ on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is analytic (see Theorem 2.4.1 in Chapter 2), the maximal regularity result [BDDM07, Theorem 3.1, p. 143] with the constraint $(I-P)\widetilde{\mathbf{z}}=0$ implies in particular that

$$\|\widetilde{\mathbf{z}}\|_{H^1(0,T;\mathbf{L}^2(\Omega))}\le C\left(\|\mathbf{F}_2\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+\|\partial_t\widetilde{\mathbf{w}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\right). \tag{3.4.31}$$

Then, from [MR10, Theorem 9.4.5] we deduce that the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{z}},\widetilde{q})=\mathbf{F}_2-\partial_t\widetilde{\mathbf{w}}-\partial_t\widetilde{\mathbf{z}}, \ \ \operatorname{div}\widetilde{\mathbf{z}}=0 \ \text{ in } Q^T, \\ \widetilde{\mathbf{z}}=0 \ \text{ on } \Sigma^T_d, \ \ \sigma(\widetilde{\mathbf{z}},\widetilde{q})\mathbf{n}=0 \ \text{ on } \Sigma^T_n, \end{cases}$$

admits a unique solution

$$\widetilde{\mathbf{z}}\in L^2(0,T;\mathbf{H}^2_\delta(\Omega)) \ \text{ and } \ \widetilde{q}\in L^2(0,T;H^1_\delta(\Omega)). \tag{3.4.32}$$

Thus, from (3.4.29), (3.4.31) and (3.4.32) we deduce that

$$\widetilde{\mathbf{v}}_2\in L^2(0,T;\mathbf{H}^2_\delta(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{p}_2\in L^2(0,T;H^1_\delta(\Omega)).$$

This completes the proof of (3.4.25).

Then, $\widetilde{\mathbf{v}}$ and $\tilde{p}$ satisfy the regularity stated in (3.4.23). Finally, using the fact that $\mathbf{v} = \widetilde{\mathbf{v}} + \mathbf{w} + \mathbf{z}$ and $p = \widetilde{p} + \pi$, and noting that the regularity of $\mathbf{w}$ and $\mathbf{z}$ is the same as that of $\widetilde{\mathbf{v}}$ (thanks to Propositions 3.4.2 and 3.4.1, respectively), and that the regularity of π is the same as that of $\widetilde{p}$ (again, thanks to Proposition 3.4.1), we deduce that

$$\begin{aligned}
&\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2})} + \|\mathbf{v}\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \|p\|_{L^2(0,T;H_\delta^{\frac{1}{2}+\alpha,1})} \\
&\qquad\qquad + \|\zeta_1\|_{H^{4,2}((0,T)\times(0,\ell_s))} + \|\zeta_2\|_{H^{2,1}((0,T)\times(0,\ell_s))} \\
&\le C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1} + \|\zeta_2^0\|_{H^1(0,\ell_s)} + \|\mathbf{F}_f\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \\
&\qquad + \|F_s\|_{L^2((0,T)\times(0,\ell_s))} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2})} + \|\partial_t \mathbf{w}\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \\
&\qquad\qquad + \|\mathbf{g}\|_{L^2(0,1;\mathbf{H}(\Gamma_i))} + \|\partial_t \mathbf{g}\|_{L^2(0,1;\mathbf{H}(\Gamma_i))}\Big).
\end{aligned} \tag{3.4.33}$$

The estimate (3.4.15) follows by using (3.4.13) in (3.4.33). This completes the proof. □

3.5 Estimates of nonlinear terms

In this section, we shall estimate the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ and $\widehat{F}_s$ in the corresponding appropriate norms. See Appendix B for the definition of $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ and $\widehat{F}_s$.

3.5.1 Auxiliary results

Lemma 3.5.1. *Let $a_0 \in (0,1/2)$. There exists a constant $C > 0$, depending only on a_0, such that, for all $0 < T < 1$ and all $\eta \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$ with $\eta(0) = 0$, the extension $\mathcal{E}\eta$ defined in (3.2.7) satisfies*

$$\begin{aligned}
\|\mathcal{E}\eta\|_{L^\infty(0,T;H^{2+a_0}(-L/2,L))} \le CT^\beta \Big(&\|\partial_t \eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\eta\|_{L^2(0,T;H^4(0,\ell_s))} \\
&+ \|\partial_t^2 \eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{3.5.1}$$

Proof. Let us first observe that from the definition given in (3.2.7) it follows that there exists a positive constant C independent of T such that

$$\|\mathcal{E}\eta(t,\cdot)\|_{H^{2+a_0}(-L/2,L)} \le C\|\eta(t,\cdot)\|_{H^{2+a_0}(0,\ell_s)} \text{ for a.e. } t \in (0,T).$$

From the last estimate we obtain

$$\|\mathcal{E}\eta\|_{L^\infty(0,T;H^{2+a_0}(-L/2,L))} \le C\|\eta\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))}. \tag{3.5.2}$$

In what follows we will prove that there exists a constant $C > 0$ independent of T such that

$$\begin{aligned}
\|\eta\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))} \le CT^\beta \Big(&\|\partial_t \eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\eta\|_{L^2(0,T;H^4(0,\ell_s))} \\
&+ \|\partial_t^2 \eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{3.5.3}$$

We split the proof in three steps.

- **Step 1.** *Interpolation and scaling arguments.*

Let us first observe that by interpolation we have

$$H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s)) \hookrightarrow H^{1-a_0/2}(0,1;H^{2+a_0}(0,\ell_s)). \tag{3.5.4}$$

Moreover, there exists a positive constant C such that

$$\|\zeta\|_{H^{1-\frac{a_0}{2}}(0,1;H^{2+a_0}(0,\ell_s))} \leq C\|\zeta\|^{1/2+a_0/4}_{L^2(0,1;H^4(0,\ell_s))}\|\zeta\|^{1/2-a_0/4}_{H^2(0,1;L^2(0,\ell_s))}, \tag{3.5.5}$$

for all $\zeta \in H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))$ satisfying $\zeta(0)=0$ and $\partial_t\zeta(0)=0$. Then, thanks to Poincaré's inequality, we have

$$\|\zeta\|_{H^{1-\frac{a_0}{2}}(0,1;H^{2+a_0}(0,\ell_s))} \leq C\|\zeta\|^{1/2+a_0/4}_{L^2(0,1;H^4(0,\ell_s))}\|\partial_t^2\zeta\|^{1/2-a_0/4}_{L^2(0,1;L^2(0,\ell_s))}. \tag{3.5.6}$$

On the other hand, from the continuous embedding $H^{1-a_0/2}(0,1)\hookrightarrow L^\infty(0,1)$, we have

$$H^{1-a_0/2}(0,1;H^{2+a_0}(0,\ell_s)) \hookrightarrow L^\infty(0,1;H^{2+a_0}(0,\ell_s)). \tag{3.5.7}$$

Moreover, there exists a positive constant C such that

$$\|\zeta\|_{L^\infty(0,1;H^{2+a_0}(0,\ell_s))} \leq C\|\zeta\|_{H^{1-\frac{a_0}{2}}(0,1;H^{2+a_0}(0,\ell_s))}, \tag{3.5.8}$$

for all $\zeta \in H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))$ satisfying $\zeta(0)=0$ and $\partial_t\zeta(0)=0$. Next, after combining the estimates (3.5.5) and (3.5.8), we get

$$\|\zeta\|_{L^\infty(0,1;H^{2+a_0}(0,\ell_s))} \leq C\|\zeta\|^{1/2+a_0/4}_{L^2(0,1;H^4(0,\ell_s))}\|\partial_t^2\zeta\|^{1/2-a_0/4}_{L^2(0,1;L^2(0,\ell_s))}, \tag{3.5.9}$$

for all $\zeta \in H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))$ satisfying $\zeta(0)=0$ and $\partial_t\zeta(0)=0$. Let $\zeta(t)=\widetilde{\zeta}(tT)$, $t\in(0,1)$. Then, since

$$\begin{aligned}
\|\zeta\|_{L^2(0,1;H^4(0,\ell_s))} &= T^{-1/2}\|\widetilde{\zeta}\|_{L^2(0,T;H^4(0,\ell_s))},\\
\|\partial_t^2\zeta\|_{L^2(0,1;L^2(0,\ell_s))} &= T^{3/2}\|\partial_t^2\widetilde{\zeta}\|_{L^2(0,T;L^2(0,\ell_s))},
\end{aligned} \tag{3.5.10}$$

from estimate (3.5.9) we obtain

$$\|\widetilde{\zeta}\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))} \leq CT^{1/2-a_0/2}\|\widetilde{\zeta}\|^{1/2+a_0/4}_{L^2(0,T;H^4(0,\ell_s))}\|\partial_t^2\widetilde{\zeta}\|^{1/2-a_0/4}_{L^2(0,T;L^2(0,\ell_s))}, \tag{3.5.11}$$

for all $\widetilde{\zeta} \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$ satisfying $\widetilde{\zeta}(0)=0$ and $\partial_t\widetilde{\zeta}(0)=0$. We highlight that the positive constant C in (3.5.11) is the same as the one appearing in (3.5.9), which is independent of T.

- **Step 2.** *Lifting.*

Before using estimate (3.5.11), we will introduce an appropriate lifting. Let us first introduce the space $H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$ endowed with the norm

$$\|\widetilde{\eta}\|_{H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))} := \left(\|\widetilde{\eta}\|^2_{L^2(0,1;H^4(0,\ell_s))} + \|\partial_t^2\widetilde{\eta}\|^2_{L^2(0,1;L^2(0,\ell_s))}\right)^{1/2}.$$

From [LM72, Theorem 3.2, p. 21 and Remark 3.3, p. 22] it follows that for the couple $(0,\partial_t\eta(0)) \in H^3_{\{0,\ell_s\}}(0,\ell_s)\times H^1_{\{0\}}(0,\ell_s)$, there exists a lifting $\widetilde{\eta}\in H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))$ such that $\widetilde{\eta}(0)=0$, $\partial_t\widetilde{\eta}(0)=T\partial_t\eta(0)$ and

$$\|\widetilde{\eta}\|_{H^{4,2}_{\{0,\ell_s\}}((0,1)\times(0,\ell_s))} \leq CT\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)}. \tag{3.5.12}$$

Let us set $\widetilde{\zeta}(t) := \eta(t) - \widehat{\eta}(t)$, where $\widehat{\eta}(t) := \widetilde{\eta}(t/T)$, $t \in (0,T)$. We observe that

$$\widetilde{\zeta} \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)), \quad \widetilde{\zeta}(0) = 0, \quad \text{and} \quad \partial_t\widetilde{\zeta}(0) = 0.$$

Then, applying estimate (3.5.11) to $\widetilde{\zeta}$, we get

$$\begin{aligned}\|\eta - \widehat{\eta}\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))} \leq CT^{1/2-a_0/2}\|\eta - \widehat{\eta}\|^{1/2+a_0/4}_{L^2(0,T;H^4(0,\ell_s))}& \\ \times\|\partial_t^2(\eta-\widehat{\eta})\|^{1/2-a_0/4}_{L^2(0,T;L^2(0,\ell_s))}&.\end{aligned} \tag{3.5.13}$$

After using

$$\begin{aligned}\|\widehat{\eta}\|_{L^2(0,T;H^4(0,\ell_s))} &= T^{1/2}\|\widetilde{\eta}\|_{L^2(0,1;H^4(0,\ell_s))},\\ \|\partial_t^2\widehat{\eta}\|_{L^2(0,T;L^2(0,\ell_s))} &= T^{-3/2}\|\partial_t^2\widetilde{\eta}\|_{L^2(0,1;L^2(0,\ell_s))},\end{aligned}$$

estimate (3.5.12), Young's inequality and the fact that $0 < T < 1$, we obtain

$$\begin{aligned}&\|\eta - \widehat{\eta}\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))}\\ &\leq CT^{\beta}\Big(\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\eta\|_{L^2(0,T;H^4(0,\ell_s))} + \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big),\end{aligned} \tag{3.5.14}$$

where $\beta = 1/4 - 3a_0/8 > 0$.

• **Step 3.** *Conclusion.*

Finally, after combining the estimates (3.5.12) and (3.5.14), [LM72, Theorem 3.1, p. 19] and using the fact that $0 < T < 1$, we obtain

$$\begin{aligned}\|\eta\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))} &\leq \|\eta - \widehat{\eta}\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))} + \|\widehat{\eta}\|_{L^\infty(0,T;H^{2+a_0}(0,\ell_s))}\\ &\leq CT^{\beta}\Big(\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\eta\|_{L^2(0,T;H^4(0,\ell_s))}\\ &\qquad + \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big),\end{aligned} \tag{3.5.15}$$

where C is a positive constant depending only on a_0. This completes the proof of (3.5.3). $\square$

Lemma 3.5.2. *Let $a_0 \in (0,1/2)$. There exists a constant $C > 0$, depending only on a_0, such that, for all $0 < T < 1$ and all $\eta \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$, the extension $\mathcal{E}\eta$ defined in (3.2.7) satisfies*

$$\begin{aligned}\|\mathcal{E}\eta\|_{L^\infty(0,T;H^2(0,\ell_s))} \leq C\Big(&\|\eta(0)\|_{H^2_{\{0,\ell_s\}}(0,\ell_s)} + \|\eta\|_{L^2(0,T;H^4(0,\ell_s))}\\ &+ \|\partial_t\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big),\end{aligned} \tag{3.5.16}$$

$$\begin{aligned}\|\partial_t\mathcal{E}\eta\|_{L^\infty(0,T;H^{a_0}(-L/2,L))} \leq C\Big(&\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\partial_t\eta\|_{L^2(0,T;H^2(0,\ell_s))}\\ &+ \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big),\end{aligned} \tag{3.5.17}$$

$$\begin{aligned}\|\partial_t\mathcal{E}\eta\|_{L^{q^*}(0,T;H^{\frac{3}{2}+a_0}(-L/2,L))} \leq C\Big(&\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\partial_t\eta\|_{L^2(0,T;H^2(0,\ell_s))}\\ &+ \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big),\end{aligned} \tag{3.5.18}$$

where $q^* = 4/(1+2a_0)$.

Moreover, if $\ell - e + \eta^{\pm}_{\chi}(t,z) \geq (\ell - e)/2$ in $(0,T)\times\Omega$, then there exists a positive constant

C, depending only on ℓ and e, such that

$$\|\det(J)\|_{L^\infty(0,T;H^2(\Omega))} \leq C\left(1+\|\eta\|_{L^2(0,T;H^2(0,\ell_s))}\right)\|\eta\|_{L^2(0,T;H^2(0,\ell_s))}, \tag{3.5.19}$$

where

$$\det(J) = \frac{\ell-e}{\ell-e+\eta_X^\pm(t,z_1)}, \quad (t,z)\in(0,T)\times\Omega.$$

Proof.

• **Proof of (3.5.16).** Let us first notice that from the definition of the extension of η given in (3.2.7), it follows that there exists a positive constant C independent of T such that

$$\|\mathcal{E}\eta(t,\cdot)\|_{H^2(-L/2,L)} \leq C\|\eta(t,\cdot)\|_{H^2(0,\ell_s)} \text{ a.e. } t\in(0,T). \tag{3.5.20}$$

Next, from these estimates we deduce that

$$\|\mathcal{E}\eta\|_{L^\infty(0,T;H^2(-L/2,L))} \leq C\|\eta\|_{L^\infty(0,T;H^2(0,\ell_s))}. \tag{3.5.21}$$

Using the estimates (3.5.21) and Lemma 3.C.3 with $X = H^4_{\{0,\ell_s\}}(0,\ell_s)$ and $Y = L^2(0,\ell_s)$, we obtain the desired esimate.

• **Proof of (3.5.17).** The proof is analogous to that of estimate (3.5.16) and is presented in Appendix D.1.

• **Proof of (3.5.18).** The proof can be adapted from that of Lemma 3.5.1. The detailed argument is presented in Appendix D.2.

• **Proof of (3.5.19).** The estimate (3.5.19) is a consequence of [MRR20, Lemma A.4]. □

Lemma 3.5.3. *Let $s_0\in(0,1/2)$. Then, for all $s\in(0,\frac{1}{2})$, there exists a constant $C>0$ such that, for all $F\in H^{-s_0}(\Omega)$ and $\widetilde{\eta}\in H^{\frac{1}{2}+s}(-L/2,L)$, with $\widetilde{\eta}=0$ on $(-L/2,0)\bigcup(\ell_s+\varepsilon/2,L)$, the following estimate holds:*

$$\|\widetilde{\eta}F\|_{H^{-s_0}(\Omega)} \leq C\|\widetilde{\eta}\|_{H^{\frac{1}{2}+s}(-L/2,L)}\|F\|_{H^{-s_0}(\Omega)}. \tag{3.5.22}$$

Proof. Since $\widetilde{\eta}=0$ on $(-L/2,0)\bigcup(\ell_s+\varepsilon/2,L)$, we assume that $\Omega=\Omega^+\cup\Omega^-$, where $\Omega^\pm = (0,\ell_s+\varepsilon/2)\times((e,\ell)\cup(-\ell,-e))\bigcup(\ell_s,\ell_s+\varepsilon/2)\times(-e,e)$. Given $\varphi\in H^{s_0}(\Omega)$, with $\|\varphi\|_{H^{s_0}(\Omega)}\leq 1$, it is sufficient to show that $\widetilde{\eta}\varphi\in H^{s_0}(\Omega^\pm)$. We will only prove that $\widetilde{\eta}\varphi\in L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))\cap H^{s_0}(0,\ell_s+\varepsilon/2;L^2(e,\ell))$. Let us first show that $\widetilde{\eta}\varphi\in L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))$. Since $\varphi(z_1,\cdot)\in H^{s_0}(e,\ell)$ for a.e. $z_1\in(0,\ell_s+\varepsilon/2)$ and $\widetilde{\eta}\in H^{\frac{1}{2}+s}(-L/2,L)\hookrightarrow L^\infty(-L/2,L)$, we have

$$\begin{aligned}
\|\eta\varphi\|^2_{L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))} &= \int_0^{\ell_s+\varepsilon/2}|\eta(z_1)|^2\|\varphi(z_1,\cdot)\|^2_{H^{s_0}(e,\ell)}\,dz_1 \\
&\leq \|\widetilde{\eta}\|^2_{L^\infty(-L/2,L)}\|\varphi\|^2_{L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))} \\
&\leq C\|\widetilde{\eta}\|^2_{H^{\frac{1}{2}+s}(-L/2,L)}\|\varphi\|^2_{L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))},
\end{aligned}$$

from where we deduce that

$$\|\eta\varphi\|_{L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))} \leq C\|\widetilde{\eta}\|_{H^{\frac{1}{2}+s}(-L/2,L)}\|\varphi\|_{L^2(0,\ell_s+\varepsilon/2;H^{s_0}(e,\ell))}. \tag{3.5.23}$$

Let us prove that $\widetilde{\eta}\varphi\in H^{s_0}(0,\ell_s+\varepsilon/2;L^2(e,\ell))$. Since $\|\widetilde{\eta}(z_1)\varphi(z_1,\cdot)\|_{L^2(e,\ell)} = |\widetilde{\eta}(z_1)|\|\varphi(z_1,\cdot)\|_{L^2(e,\ell)}$

for a.e. $z_1 \in (0, \ell_s + \varepsilon/2)$ and, $z_1 \mapsto \widetilde{\eta}(z_1) \in H^{\frac{1}{2}+s}(0, \ell_s + \varepsilon/2)$, $z_1 \mapsto \|\varphi(z_1, \cdot)\|_{L^2(e,\ell)} \in H^{s_0}(0, \ell_s + \varepsilon/2)$, from [GS91, Proposition B.1] we have that

$$\|\widetilde{\eta}\varphi\|_{H^{s_0}(0,\ell_s+\varepsilon/2;L^2(e,\ell))} \le C\|\widetilde{\eta}\|_{H^{\frac{1}{2}+s}(-L/2,L)}\|\varphi\|_{H^{s_0}(0,\ell_s+\varepsilon/2;L^2(e,\ell))}. \tag{3.5.24}$$

From (3.5.23) and (3.5.24), we deduce that $\widetilde{\eta}\varphi \in L^2(0, \ell_s+\varepsilon/2; H^{s_0}(e, \ell)) \cap H^{s_0}(0, \ell_s+\varepsilon/2; L^2(e, \ell))$. □

Lemma 3.5.4. *There exists a positive constant C, depending only on $\alpha \in (0, \alpha^*)$, $s \in (1/2, 1/2+\alpha)$, such that for all $0 < T < 1$ and all $\mathbf{v} \in L^\infty(0, T; \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)) \cap L^2(0, T; \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega))$, we have*

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{1+s}(\Omega))} \le CT^{\frac{1}{2}\left(\frac{1}{4}-\frac{s}{2}+\frac{\alpha}{2}\right)}\|\mathbf{v}\|^{\frac{1}{4}-\frac{s}{2}+\frac{\alpha}{2}}_{L^\infty(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))}\|\mathbf{v}\|^{\frac{3}{4}+\frac{s}{4}-\frac{\alpha}{4}}_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega))}, \tag{3.5.25}$$

$$\begin{aligned}\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega))} \le C\Big(&\|\mathbf{v}(0)\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)} + \|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega))} \\ &+ \|\partial_t \mathbf{v}\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))}\Big).\end{aligned} \tag{3.5.26}$$

Proof.
• **Proof of (3.5.25)**. Let us first notice that by interpolation we have that there exists a constant $C > 0$ independent of T such that

$$\|\mathbf{v}(t,\cdot)\|_{\mathbf{H}^{1+s}(\Omega)} \le C\|\mathbf{v}(t,\cdot)\|^{\frac{3}{4}+\frac{s}{2}-\frac{\alpha}{2}}_{\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)}\|\mathbf{v}(t,\cdot)\|^{\frac{1}{4}-\frac{s}{2}+\frac{\alpha}{2}}_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)}.$$

Next, using Hölder's inequality with $p = 1/\left(3/4 + s/2 - \alpha/2\right)$ and $p' = 1/\left(1/4 - s/2 + \alpha/2\right)$, we get

$$\|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^{1+s}(\Omega))} \le CT^{\left(\frac{1}{4}-\frac{s}{2}+\frac{\alpha}{2}\right)}\|\mathbf{v}\|^{2\left(\frac{1}{4}-\frac{s}{2}+\frac{\alpha}{2}\right)}_{L^\infty(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))}\left(\int_0^T \|\mathbf{v}(t,\cdot)\|^2_{\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)}\,dt\right)^{\frac{3}{4}+\frac{s}{2}-\frac{\alpha}{2}}.$$

This proves the estimate (3.5.25).

• **Proof of (3.5.26)**. It is a consequence of Lemma 3.C.3 with $X = \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and $Y = \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)$. □

3.5.2 Estimates of nonlinear terms

This subsection is devoted to the estimate of the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ and $\widehat{F}_s$.

Let $R > 0$ and let $a_0 \in (0, 1/2)$ be the parameter that appears in Proposition 3.2.2. Let us also assume that $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\eta_2^0 \in H^1_{\{0\}}(0, \ell_s)$. Let us recall that the set $B(T, R, \mathbf{u}_0, \eta_2^0)$ is defined by

$$\begin{aligned}B(T, R, \mathbf{u}_0, \eta_2^0) := \Big\{&(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in \mathcal{Z}_T \mid \|(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{\mathcal{Z}_T} \le R,\ \ \eta \in E(0, T) \\ &\text{and } \widehat{\mathbf{u}}(0) = \mathbf{u}_0,\ \eta(0) = 0,\ \eta_t(0) = \eta_2^0\Big\},\end{aligned} \tag{3.5.27}$$

where the space $\mathcal{Z}_T$ is defined in (3.2.18).

Estimate of $\widehat{\mathbf{F}}_f$

Lemma 3.5.5. *There exist constants $C_{\widehat{\mathbf{F}}_f} > 0$ and $\beta > 0$, depending only on R, a_0, $\mathbf{u}_0$ and η_2^0, such that, for all $0 < T < 1$, all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have*

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \leq C_{\widehat{\mathbf{F}}_f} T^\beta. \tag{3.5.28}$$

Furthermore, for all $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1), (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have

$$\begin{aligned} &\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - \widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ &\qquad\qquad \leq C_{\widehat{\mathbf{F}}_f} T^\beta \|(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{\mathcal{Z}_T}. \end{aligned} \tag{3.5.29}$$

Proof. We recall that $\widehat{\mathbf{F}}_f$ is given in Appendix B, (3.6.21).

Firstly, given the similarity between some of the terms present in $\widehat{\mathbf{F}}_f$, we will only show the estimates (3.5.28) and (3.5.29) for some of them. In particular, we have selected the following terms:

- $\widetilde{\mathbf{F}}_f^1 := \det(J)(\partial_{z_1}\eta^\pm)\dfrac{\partial \widehat{u}_i}{\partial z_1}\widehat{u}_1$,
- $\widetilde{\mathbf{F}}_f^2 := \det(J)(\eta_t^\pm)\dfrac{\partial \widehat{u}_i}{\partial z_2}$,
- $\widetilde{\mathbf{F}}_f^3 := \det(J)(\partial_{z_1}\eta^\pm)\dfrac{\partial^2 \widehat{u}_i}{\partial z_2 \partial z_1}$,
- $\widetilde{\mathbf{F}}_f^4 := \det(J)(\partial_{z_1}^2\eta^\pm)\dfrac{\partial \widehat{u}_i}{\partial z_2}$,

where

$$\eta^+(t,\cdot,e) := (\mathcal{E}\eta)(t,\cdot) \text{ on } [-L/2, L], \quad \eta^-(t,\cdot,-e) := (\mathcal{E}\eta)(t,\cdot) \text{ on } [-L/2, L],$$

and $\det(J) = \frac{\ell - e}{\ell - e + \eta_\chi^\pm(t,z)}$, with $\eta_\chi^\pm(t,z) := (\mp\chi^\pm + (\ell \mp z_2)\partial_{z_2}\chi^\pm)\,\eta^\pm(t,z_1)$, $(t,z) \in [0,T] \times \overline{\Omega}$. Secondly, let us observe that in order to show the estimate (3.5.28), it is sufficient to prove

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha})} \leq C_{\widehat{\mathbf{F}}_f} T^\beta \tag{3.5.30}$$

and

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}, \widehat{p}, \eta)|_{\Omega_2}\|_{L^2(0,T;\mathbf{L}^2(\Omega_2))} \leq C_{\widehat{\mathbf{F}}_f} T^\beta. \tag{3.5.31}$$

The Lipschitz estimate (3.5.29) can be obtained in a similar way to how we will proceed to prove (3.5.28).

- $\widetilde{\mathbf{F}}_f^1$: By using the Hölder inequality, the continuous embedding $H^s(\Omega) \hookrightarrow L^4(\Omega)$ with $1/2 < s \leq 1/2 + \alpha$ and Lemmas 3.5.1, 3.5.2 and 3.5.4, we obtain

$$\begin{aligned} &\left\|\det(J)(\partial_{z_1}\eta^\pm)\frac{\partial \widehat{u}_i}{\partial z_1}\widehat{u}_1\right\|_{L^2(0,T;L^2(\Omega))} \\ &\leq C\,\|\det(J)\|_{L^\infty((0,T)\times\Omega)}\,\|\partial_{z_1}\eta^\pm\|_{L^\infty((0,T)\times(-L/2,L))} \\ &\qquad\qquad \times \|\widehat{u}_1\|_{L^\infty(0,T;H^{\frac{1}{2}+\alpha}(\Omega))} \times \left\|\frac{\partial \widehat{u}_i}{\partial z_1}\right\|_{L^2(0,T;H^s(\Omega))} \\ &\leq CT^\beta. \end{aligned}$$

- $\widetilde{\mathbf{F}}_f^2$: Let $a_0 \in (1/4, 1/2)$ be the parameter that appears in Proposition 3.2.2. By using Hölder's inequality and the continuous embeddings $H^{a_0}(-L/2, L) \hookrightarrow L^4(-L/2, L)$ and $H^s(\Omega) \hookrightarrow L^4(\Omega)$, with $1/2 < s < 1/2 + \alpha$, together with Lemmas 3.5.2 and 3.5.4, we obtain

$$\begin{aligned}\left\|\det(J)\eta_t^{\pm}\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(0,T;L^2(\Omega))} &\le C\,\|\det(J)\|_{L^\infty((0,T)\times\Omega)} \\ &\qquad \times \|\eta_t^{\pm}\|_{L^\infty(0,T;H^{a_0}(-L/2,L))}\left\|\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(0,T;H^s(\Omega))} \\ &\le CT^\beta\end{aligned}$$

- $\widetilde{\mathbf{F}}_f^3$: From Lemmas 3.5.1, 3.5.2 and 3.5.3, we get

$$\begin{aligned}&\left\|\det(J)(\partial_{z_1}\eta^{\pm})\frac{\partial^2 \widehat{u}_i}{\partial z_2 \partial z_1}\right\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha}(\Omega))} \\ &\le C\,\|\det(J)\|_{L^\infty((0,T)\times\Omega)}\,\|\partial_{z_1}\eta^{\pm}\|_{L^\infty(0,T;H^{\frac{1}{2}+a_0}(-L/2,L))} \\ &\qquad \times \left\|\frac{\partial^2 \widehat{u}_i}{\partial z_2 \partial z_1}\right\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha_0}(\Omega))} \\ &\le CT^\beta.\end{aligned}$$

- $\widetilde{\mathbf{F}}_f^4$: Let $a_0 \in (1/4, 1/2)$ be the parameter that appears in Proposition 3.2.2. Using the Hölder inequality and the continuous embeddings $H^{a_0}(-L/2,L) \hookrightarrow L^4(-L/2,L)$ and $H^s(\Omega) \hookrightarrow L^4(\Omega)$, with $1/2 < s < 1/2 + \alpha$, together with Lemmas 3.5.1, 3.5.2 and 3.5.4, we obtain

$$\begin{aligned}&\left\|\det(J)(\partial_{z_1}^2\eta^{\pm})\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(0,T;L^2(\Omega))} \\ &\le C\,\|\det(J)\|_{L^\infty((0,T)\times\Omega)}\,\|\partial_{z_1}^2\eta^{\pm}\|_{L^\infty(0,T;H^{a_0}(-L/2,L))}\left\|\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(0,T;H^s(\Omega))} \\ &\le CT^\beta.\end{aligned}$$

This completes the proof. □

Estimate of $\widehat{\mathbf{G}}_{\mathrm{div}}$

Lemma 3.5.6. *There exist constants $C_{\widehat{\mathbf{G}}_{\mathrm{div}}} > 0$ and $\beta > 0$, depending only on R, a_0, $\mathbf{u}_0$ and η_2^0, such that, for all $0 < T < 1$, and for all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have*

$$\begin{aligned}\left\|\mathrm{div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}\right)(\widehat{\mathbf{u}},\eta)\right\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}},\eta)\|_{H^1(0,T;H^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ \le C_{\widehat{\mathbf{G}}_{\mathrm{div}}} T^\beta\end{aligned} \tag{3.5.32}$$

and

$$\left\|(1-\widetilde{\theta})\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}},\eta)\right\|_{L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \le C_{\widehat{\mathbf{G}}_{\mathrm{div}}} T^\beta, \tag{3.5.33}$$

where $\widetilde{\theta}$ is introduced in (3.4.7). Furthermore, for all $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1)$ and $(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have

$$\begin{aligned}&\left\|\left(\mathrm{div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}\right)(\widehat{\mathbf{u}}^1,\eta^1) - \mathrm{div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}\right)(\widehat{\mathbf{u}}^2,\eta^2)\right)\right\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1}(\Omega))} \\ &+ \left\|\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}^1,\eta^1) - \widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}^2,\eta^2)\right\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ &+ \left\|(1-\Psi)\left(\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}^1,\eta^1) - \widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}^2,\eta^2)\right)\right\|_{L^2(0,T;\mathbf{H}_{2,\delta}^{\frac{3}{2}+\alpha}(\Omega))\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ &\le C_{\widehat{\mathbf{G}}_{\mathrm{div}}} T^\beta \|(\widehat{\mathbf{u}}^1,\widehat{p}^1,\eta^1) - (\widehat{\mathbf{u}}^2,\widehat{p}^2,\eta^2)\|_{\mathcal{Z}_T}.\end{aligned} \tag{3.5.34}$$

Proof. We recall that $\widehat{\mathbf{G}}_{\rm div}$ is given in Appendix B, (3.6.22). We will only present the proofs of estimates (3.5.32) and (3.5.33). The proof of the Lipschitz estimate (3.5.34) can be proved in a similar way.

Let us show the estimates (3.5.32) and (3.5.33). We split the proof in three steps.

- Step 1: *Proof of estimate* $\left\|\mathrm{div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}\right)(\widehat{\mathbf{u}},\eta)\right\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1})} \leq C_{\widehat{\mathbf{G}}_{\rm div}} T^\beta$.

We will only consider the term $(\partial_{z_1}\eta^\pm)\frac{\partial \widehat{u}_1}{\partial z_2}$. In this case, the proof of

$$\left\|(\partial_{z_1}\eta^\pm)\frac{\partial \widehat{u}_1}{\partial z_2}\right\|_{L^2(0,T;L^2)} + \left\|\nabla\left((\partial_{z_1}\eta^\pm)\frac{\partial \widehat{u}_1}{\partial z_2}\right)\right\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \leq CT^\beta \tag{3.5.35}$$

may be adapted from the one of Lemma 3.5.5.

- Step 2: *Proof of estimate* $\|\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}},\eta)\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \leq C_{\widehat{\mathbf{G}}_{\rm div}} T^\beta$.

We will prove that

$$\|(\partial_{z_1}\eta^\pm)\widehat{u}_1\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} \leq CT^\beta \tag{3.5.36}$$

and

$$\|\partial_t((\partial_{z_1}\eta^\pm)\widehat{u}_1)\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} \leq CT^\beta. \tag{3.5.37}$$

To prove estimate (3.5.36), we can proceed in a similar way as in the proof of Lemma 3.5.5.

In order to prove (3.5.37), we will show that

$$\|(\partial^2_{z_1 t}\eta^\pm)\widehat{u}_1\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} \leq CT^\beta \tag{3.5.38}$$

and

$$\|(\partial_{z_1}\eta^\pm)\partial_t\widehat{u}_1\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} \leq CT^\beta. \tag{3.5.39}$$

Let us begin by proving (3.5.38). Let us first observe that thanks to the embedding $H^{\frac{1}{2}+\alpha}(\Omega) \hookrightarrow L^2(\Omega)$, Lemmas 3.5.2 and 3.5.4, we obtain the following estimate:

$$\left\|(\partial^2_{z_1 t}\eta^\pm)\widehat{u}_1\right\|_{L^2(0,T;L^2(\Omega))} \leq C\|\partial^2_{z_1 t}\eta^\pm\|_{L^2(0,T;L^\infty(-L/2,L))}. \tag{3.5.40}$$

To conclude the proof of (3.5.38), we will show that

$$\|\partial^2_{z_1 t}\eta^\pm\|_{L^2(0,T;L^\infty(-L/2,L))} \leq CT^\beta. \tag{3.5.41}$$

Let us notice that for all $a_0 \in (0,1/2)$ we have that $H^{1/4-a_0/2}(0,T) \hookrightarrow L^{q^*}(0,T)$, with $q^* = 4/(1+2a_0)$. We highlight that the continuity constant in the preceding embedding does not depend on T (see [DPV12, Remark 6.8]). Then, by using Hölder's inequality and Lemma 3.5.2, we obtain

$$\int_0^T \|\partial^2_{z_1 t}\eta^\pm\|^2_{L^\infty(-L/2,L)}\,dt \leq T^{\frac{q^*}{q^*-2}}\|\partial^2_{z_1 t}\eta^\pm\|^2_{L^{q^*}(0,T;L^\infty(-L/2,L))} \leq CT^{\frac{q^*}{q^*-2}}. \tag{3.5.42}$$

This proves (3.5.41) and hence the estimate (3.5.38).

Let us now prove (3.5.39). With Lemmas 3.5.1 and 3.5.3, we have the following estimate:

$$\begin{aligned}&\|\partial_{z_1}\eta^{\pm}\partial_t\widehat{u}_1\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha}(\Omega))}\\&\leq C\|\partial_{z_1}\eta^{\pm}\|_{L^\infty(0,T;H^{\frac{1}{2}+a_0}(-L/2,L))}\|\partial_t\widehat{u}_1\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha}(\Omega))}\\&\leq CT^\beta.\end{aligned}$$

Thus, the estimate (3.5.32) follows from steps 1 and 2.

• Step 3: *Proof of estimate* (3.5.33).

Let $a_0 \in (0,1/2)$ be the parameter that appears in Proposition 3.2.2. Since $\eta^{\pm}(t,\cdot) \in H^4(\epsilon_0, L) \hookrightarrow L^\infty(\varepsilon_0, L)$, from Lemma 3.5.1 we obtain

$$\begin{aligned}\|(1-\widetilde{\theta})(\partial_{z_1}\eta^{\pm})\widehat{u}_1\|_{L^2(0,T;H^{\frac{3}{2}+\alpha}(\Omega))} &\leq C\|\partial_{z_1}\eta^{\pm}\|_{L^\infty(0,T;H^{\frac{1}{2}+a_0}(\varepsilon_0,L))}\|\widehat{u}_1\|_{L^2(0,T;H^{\frac{3}{2}+\alpha}(\Omega))}\\&\leq CT^\beta.\end{aligned}\tag{3.5.43}$$

This completes the proof. □

Estimate of $\widehat{F}_s$

Lemma 3.5.7. *There exist constants $C_{\widehat{F}_s} > 0$ and $\beta > 0$, depending only on R, a_0, $\mathbf{u}_0$ and η_2^0, such that, for all $0 < T < 1$, and for all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have*

$$\|\widehat{F}_s(\widehat{\mathbf{u}}, \eta)\|_{L^2(0,T;H^\alpha(0,\ell_s))} \leq C_{\widehat{F}_s}T^\beta.\tag{3.5.44}$$

Furthermore, for all $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1), (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, we have

$$\begin{aligned}&\|\widehat{F}_s(\widehat{\mathbf{u}}^1, \eta^1) - \widehat{F}_s(\widehat{\mathbf{u}}^2, \eta^2)\|_{L^2(0,T;H^\alpha(0,\ell_s))}\\&\qquad\qquad\leq C_{\widehat{F}_s}T^\beta\|(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{\mathcal{Z}_T}.\end{aligned}\tag{3.5.45}$$

Proof. We first recall that $\widehat{F}_s$ is given in Appendix B, (3.6.23). Analogously to the proof of the Lemmas 3.5.5 and 3.5.6, we will only prove the estimates (3.5.44) and (3.5.45) for some terms of $\widehat{F}_s$. In this particular case, we will only consider the term

$$\gamma_s^+\left(\det(J)(\partial_{z_1}\eta^{\pm})\frac{\partial\widehat{u}_2}{\partial z_2}\right).$$

Let us remark that in order to prove the Lipschitz estimate (3.5.45) we can proceed in a similar manner as we will proceed to show (3.5.44).

Since $\gamma_s^+ \in \mathcal{L}(H^{\frac{1}{2}+\alpha}(\Omega), H^\alpha(\Gamma_s^+))$, it suffices to show that

$$\left\|\det(J)(\partial_{z_1}\eta^{\pm})\frac{\partial\widehat{u}_2}{\partial z_2}\right\|_{L^2(0,T;H^{\frac{1}{2}+\alpha})} \leq CT^\beta.\tag{3.5.46}$$

To conclude the estimate (3.5.46) we will prove that

$$\left\|\det(J)(\partial_{z_1}\eta^{\pm})\frac{\partial\widehat{u}_2}{\partial z_2}\right\|_{L^2(0,T;L^2)} \leq CT^\beta\tag{3.5.47}$$

and

$$\left\|\nabla\left(\det(J)(\partial_{z_1}\eta^{\pm})\frac{\partial\widehat{u}_2}{\partial z_2}\right)\right\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha})} \leq CT^\beta.\tag{3.5.48}$$

Let us first prove the estimate (3.5.47). Let $a_0 \in (1/4, 1/2)$ be the parameter that appears in Proposition 3.2.2. Using Hölder's inequality and the continuous embeddings $H^{a_0}(-L/2, L) \hookrightarrow L^4(-L/2, L)$ and $H^s(\Omega) \hookrightarrow L^4(\Omega)$, with $1/2 < s < 1/2 + \alpha$, with Lemmas 3.5.1, 3.5.2 and 3.5.4, we get

$$\begin{aligned}
&\left\| \det(J)(\partial_{z_1}\eta^{\pm}) \frac{\partial \widehat{u}_1}{\partial z_2} \right\|_{L^2(0,T;L^2(\Omega))} \\
&\leq C \left\| \det(J) \right\|_{L^\infty((0,T)\times\Omega)} \left\| \partial_{z_1}\eta^{\pm} \right\|_{L^\infty(0,T;H^{a_0}(-L/2,L))} \left\| \frac{\partial \widehat{u}_1}{\partial z_2} \right\|_{L^2(0,T;H^s(\Omega))} \\
&\leq CT^\beta.
\end{aligned}$$

This completes the proof of (3.5.47).

Let us now show (3.5.48). We will only consider the estimate

$$\left\| \det(J)(\partial^2_{z_1}\eta^{\pm}) \frac{\partial \widehat{u}_1}{\partial z_2} \right\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} + \left\| \det(J)\partial_{z_1}\eta^{\pm} \frac{\partial^2 \widehat{u}_1}{\partial z_2^2} \right\|_{L^2(0,T;H^{-\frac{1}{2}+\alpha})} \leq CT^\beta. \tag{3.5.49}$$

The proof of (3.5.49) is similar of that of Lemma 3.5.5. □

3.6 Proof of Theorem 3.2.1

Let us first recall the definition of the space $\mathcal{Z}_T$ given in (3.2.18):

$$\begin{aligned}
\mathcal{Z}_T = \Big(L^2(0,T;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)) \cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \Big) \\
\times L^2(0,T;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)).
\end{aligned}$$

We now introduce the mapping $\mathcal{N}$ from $\mathcal{Z}_T$ into itself, defined by

$$\mathcal{N}(\mathbf{\Phi}, \psi, k) = (\widehat{\mathbf{u}}, \widehat{p}, \eta) \quad \text{for all } (\mathbf{\Phi}, \psi, k) \in \mathcal{Z}_T,$$

where $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ is the solution of the system

$$\begin{cases}
\partial_t \widehat{\mathbf{u}} - \operatorname{div}\sigma(\widehat{\mathbf{u}}, \widehat{p}) = \widehat{\mathbf{F}}_f(\mathbf{\Phi}, \psi, k), \ \ \operatorname{div}\widehat{\mathbf{u}} = \operatorname{div}\widehat{\mathbf{G}}_{\mathrm{div}}(\mathbf{\Phi}, k) \ \text{ in } Q^T, \\
\widehat{\mathbf{u}} = \mathbf{g}^i \ \text{ on } \Sigma^T_i, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma^T_w \cup \Sigma^T_r, \ \ \widehat{\mathbf{u}} = \eta_t \vec{\mathbf{e}}_2 \ \text{ on } \Sigma^T_s, \\
\sigma(\widehat{\mathbf{u}}, \widehat{p})\mathbf{n} = 0 \ \text{ on } \Sigma^T_n, \ \ \widehat{\mathbf{u}}(0) = \mathbf{u}_0(X_\eta(\cdot)) \ \text{ in } \Omega, \\
\partial^2_t \eta_t + \alpha\Delta^2_s \eta + \gamma(\Delta^2_s)^{\frac{1}{2}}\eta_t = -\gamma^+_s \widehat{p} + \gamma^-_s \widehat{p} + \widehat{F}_s(\mathbf{\Phi}, k) \ \text{ in } (0,T)\times(0,\ell_s), \\
\eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ on } (0,T)\times\{0\}, \\
\partial^2_{x_1}\eta = 0 \ \text{ and } \ \partial^3_{x_1}\eta = 0 \ \text{ on } (0,T)\times\{\ell_s\}, \\
\eta(0) = 0 \ \text{ and } \ \eta_t(0) = \eta^0_2 \ \text{ in } (0,\ell_s).
\end{cases} \tag{3.6.1}$$

To prove the existence of a solution to the system (3.6.1), we will show that for an appropriate radius $R > 0$, there exists $T \in (0,1)$ sufficiently small such that the mapping $\mathcal{N}$ maps $B(T, R, \mathbf{u}_0, \eta^0_2)$ into $B(T, R, \mathbf{u}_0, \eta^0_2)$ and it is a strict contraction in $B(T, R, \mathbf{u}_0, \eta^0_2)$.

Proposition 3.6.1. *Let M be a positive constant satisfying*

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\eta^0_2\|_{H^1(0,\ell_s)} + \|\mathbf{g}^i\|_{H^1_{\{0\}}(0,1;\mathbf{H}(\Gamma_i))} \leq M. \tag{3.6.2}$$

Then, there exists $T \in (0,1)$ such that $\mathcal{N}$ maps $B(T, R, \mathbf{u}_0, \eta^0_2)$ into itself, where $R = 2C_L M$, with C_L being the constant appearing in estimate (3.4.15). Moreover, for all $(\mathbf{\Phi}, \psi, k) \in B(T, R, \mathbf{u}_0, \eta^0_2)$, *all* $(\mathbf{\Phi}^1, \psi^1, k^1)$, $(\mathbf{\Phi}^2, \psi^2, k^2) \in B(T, R, \mathbf{u}_0, \eta_2^0)$, *we have*

$$\|\mathcal{N}(\mathbf{\Phi}, \psi, k)\|_{\mathcal{Z}_T} \leq R \tag{3.6.3}$$

and

$$\|\mathcal{N}(\mathbf{\Phi}^1, \psi^1, k^1) \; - \mathcal{N}(\mathbf{\Phi}^2, \psi^2, k^2)\|_{\mathcal{Z}_T} \leq CT^\beta \|(\mathbf{\Phi}^1, \psi^1, k^1) - (\mathbf{\Phi}^2, \psi^2, k^2)\|_{\mathcal{Z}_T}. \tag{3.6.4}$$

Proof. From Theorem 3.4.1 it follows that

$$\begin{aligned}
&\|(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{\mathcal{Z}_T} \leq C_L \Big(\|\mathbf{u}_0\|_{\mathbf{H}^1} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\mathbf{g}\|_{H^1(0,T;\mathbf{H}(\Gamma_i))} \\
&+ \|\widehat{\mathbf{F}}_f(\mathbf{\Phi}, \psi, k)\|_{L^2(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \|\widehat{F}_s(\mathbf{\Phi}, k)\|_{L^2((0,T)\times(0,\ell_s))} \\
&+ \left\| \operatorname{div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\mathbf{\Phi}, k)\right)\right\|_{L^2(0,T;H^{\frac{1}{2}+\alpha,1})} + \|\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\mathbf{\Phi}, k)\|_{H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \\
&+ \left\|(1-\widetilde{\theta})\widehat{\mathbf{G}}_{\mathrm{div}}(\mathbf{\Phi}, k)\right\|_{L^2(0,T;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2})\cap H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \Big),
\end{aligned} \tag{3.6.5}$$

where $C_L > 0$ is the constant appearing in estimate (3.4.15), which is independent of T. After combining the estimates (3.5.28), (3.5.32), (3.5.33), (3.5.44), (3.6.2), in (3.6.5), we get

$$\|(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{\mathcal{Z}_T} \leq C_L M + C_L (C_{\widehat{\mathbf{F}}_f} + C_{\widehat{\mathbf{G}}_{\mathrm{div}}} + C_{\widehat{F}_s}) T^\beta.$$

We recall that M depends on the initial conditions $\mathbf{u}_0, \eta_2^0$ and on the boundary data $\mathbf{g}^i$. Next, for $T \in (0,1)$ small enough we deduce that

$$\|(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{\mathcal{Z}_T} \leq R.$$

From Lemma 3.5.1, we get

$$\|\eta^\pm\|_{L^\infty((0,T)\times(-L/2,L))} \leq CT^\beta,$$

with $C > 0$ independent of T. Then, by choosing $T > 0$ sufficiently small, we obtain that $\ell - e + \eta_\chi^\pm \geq (\ell - e)/2$ for all $(t,z) \in (0,T) \times \Omega$. Therefore, we have that $\mathcal{N}(B(T, R, \mathbf{u}_0, \eta_2^0)) \subset B(T, R, \mathbf{u}_0, \eta_2^0)$. Now, we are going to show that $\mathcal{N}$ is a strict contraction in $B(T, R, \mathbf{u}_0, \eta_2^0)$. For $j = 1, 2$, we set $\mathcal{N}(\mathbf{\Phi}^j, \psi^j, k^j) = (\widehat{\mathbf{u}}^j, \widehat{p}^j, \eta^j)$. From Theorem 3.4.1 and the estimates (3.5.29), (3.5.34) and (3.5.45), we obtain

$$\begin{aligned}
&\|(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{\mathcal{Z}_T} \\
&\qquad \leq C_L (C_{\widehat{\mathbf{F}}_f} + C_{\widehat{\mathbf{G}}_{\mathrm{div}}} + C_{\widehat{F}_s}) T^\beta \|(\mathbf{\Phi}^1, \psi^1, k^1) - (\mathbf{\Phi}^2, \psi^2, k^2)\|_{\mathcal{Z}_T}.
\end{aligned}$$

Then, if $0 < T < 1$ is small enough, we conclude that $\mathcal{N}$ is a strict contraction in $B(T, R, \mathbf{u}_0, \eta_2^0)$. $\square$

Proof of Theorem 3.2.1. The first part of Theorem 3.2.1 follows from Proposition 3.6.1 and the Banach fixed point Theorem. On the other hand, since $X(t,\cdot)$ is a $\mathcal{C}^1-$diffeomorphism from Ω into $\Omega_{\eta(t)}$, there exists a unique $Y(t,\cdot)$ from $\Omega_{\eta(t)}$ into Ω such that $X(t,\cdot)^{-1} = Y(t,\cdot)$. Defining $\mathbf{u}(t,x) = \widehat{\mathbf{u}}(t, Y(t,x))$ and $p(t,x) = \widehat{p}(t, Y(t,x))$ we can verify that $(\mathbf{u}, p, \eta)$ satisfies the system (3.1.1)-(3.1.2).

Appendix A: Expression of the function H and simplification in the modelling

In this section, we present the formal computations leading to the expression of the function $H = H(\mathbf{u}, p, \eta)$ in the system

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, p) = 0 \text{ in } Q_\eta^T, \tag{3.6.6a}$$
$$\operatorname{div} \mathbf{u} = 0 \text{ in } Q_\eta^T, \tag{3.6.6b}$$
$$\mathbf{u} = \mathbf{g}^i \text{ on } \Sigma_i^T, \quad \mathbf{u} = 0 \text{ on } \Sigma_w^T, \quad \mathbf{u} = 0 \text{ on } \Sigma_r^T, \tag{3.6.6c}$$
$$\mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \text{ on } \Sigma_\eta^T, \quad \sigma(\mathbf{u}, p)\mathbf{n} = 0 \text{ on } \Sigma_n^T, \tag{3.6.6d}$$
$$\mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega, \tag{3.6.6e}$$
$$\partial_t^2 \eta + \alpha \Delta_s^2 \eta + \gamma B \eta_t = H(\mathbf{u}, p, \eta) \text{ in } (0, T) \times (0, \ell_s), \tag{3.6.6f}$$
$$\eta = 0 \text{ and } \partial_{x_1} \eta = 0 \text{ on } (0, T) \times \{0\}, \tag{3.6.6g}$$
$$\partial_{x_1}^2 \eta = 0 \text{ and } \partial_{x_1}^3 \eta = 0 \text{ on } (0, T) \times \{\ell_s\}, \tag{3.6.6h}$$
$$\eta(0) = 0 \text{ and } \partial_t \eta(0) = \eta_2^0 \text{ in } (0, \ell_s). \tag{3.6.6i}$$

The computations rely on an energy equality. For simplicity, we assume that $\mathbf{g}^i = 0$ on Γ_i and that $B = \Delta_s^2$, with $\mathcal{D}(B) = H^4_{\{0,\ell_s\}}(0, \ell_s)$.

Let us first derive an energy identity for the sub-fluid system (3.6.6a)-(3.6.6e). By taking the scalar product of both sides of the momentum equation (3.6.6a) with the fluid velocity $\mathbf{u}$, and then integrating over $\Omega_{\eta(t)}$, we obtain

$$\int_{\Omega_{\eta(t)}} \partial_t \mathbf{u} \cdot \mathbf{u} + \int_{\Omega_{\eta(t)}} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} + 2\nu \int_{\Omega_{\eta(t)}} |\varepsilon(\mathbf{u})|^2 - \int_{\partial\Omega_{\eta(t)}} \sigma(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{u} = 0. \tag{3.6.7}$$

The first term in the previous identity can be rewritten with the aid of the Reynold's transport Theorem. Indeed, using the cited theorem (see [BF12, Theorem I.2.1]), we get

$$\frac{d}{dt} \int_{\Omega_{\eta(t)}} |\mathbf{u}|^2 = \int_{\Omega_{\eta(t)}} \partial_t(|\mathbf{u}|^2) + \int_{\Gamma_{\eta(t)}} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n},$$

from where we deduce that

$$\int_{\Omega_{\eta(t)}} \partial_t \mathbf{u} \cdot \mathbf{u} = \frac{1}{2} \frac{d}{dt} \int_{\Omega_{\eta(t)}} |\mathbf{u}|^2 - \frac{1}{2} \int_{\Gamma_{\eta(t)}} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}. \tag{3.6.8}$$

Let us analyze the second term in the identity (3.6.7). From the divergence's Theorem and the incompressbility condition $\operatorname{div} \mathbf{u} = 0$ in $\Omega_{\eta(t)}$ it follows that

$$\int_{\Omega_{\eta(t)}} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} = \frac{1}{2} \int_{\Gamma_{\eta(t)}} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n} + \frac{1}{2} \int_{\Gamma_n} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}. \tag{3.6.9}$$

Now, using identities (3.6.8) and (3.6.9) in (3.6.7), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_{\eta(t)}} |\mathbf{u}|^2 + \frac{1}{2} \int_{\Gamma_n} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n} + 2\nu \int_{\Omega_{\eta(t)}} |\varepsilon(\mathbf{u})|^2 - \int_{\partial\Omega_{\eta(t)}} \sigma(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{u} = 0. \tag{3.6.10}$$

Let us now derive an energy identity for the sub-structure system (3.6.6f)-(3.6.6i). By multiplying both sides of the identity (3.6.6f) by $\partial_t \eta$, integrating over $(0, \ell_s)$ and some integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^{\ell_s}\left(|\partial_t\eta|^2+\alpha|\eta_{xx}|^2\right)+\gamma\int_0^{\ell_s}|\eta_{txx}|^2=\int_0^{\ell_s}H(\mathbf{u},p,\eta)\partial_t\eta. \tag{3.6.11}$$

Now, after adding identities (3.6.10) and (3.6.11), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\eta(t)}}|\mathbf{u}|^2+\frac{1}{2}\int_{\Gamma_n}|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}+2\nu\int_{\Omega_{\eta(t)}}|\varepsilon(\mathbf{u})|^2\\&+\frac{1}{2}\frac{d}{dt}\int_0^{\ell_s}\left(|\partial_t\eta|^2+\alpha|\eta_{xx}|^2\right)+\gamma\int_0^{\ell_s}|\eta_{txx}|^2-\int_{\partial\Omega_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{u}=\int_0^{\ell_s}H(\mathbf{u},p,\eta)\partial_t\eta.\end{aligned} \tag{3.6.12}$$

This last identity suggests defining the function H such that

$$\int_0^{\ell_s}H(\mathbf{u},p,\eta)\partial_t\eta=-\int_{\partial\Omega_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{u}.$$

Let us analyze the right-hand side of the last equality. Indeed, let us first observe that

$$-\int_{\partial\Omega_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{u}=\underbrace{-\int_{\Gamma^+_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}^+_{\eta(t)}\cdot\mathbf{u}}_{J_1}\underbrace{-\int_{\Gamma^-_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\cdot\mathbf{u}}_{J_2}\underbrace{-\int_{\Gamma^\ell_{\eta(t)}}\sigma(\mathbf{u},p)\mathbf{n}^\ell_{\eta(t)}\cdot\mathbf{u}}_{J_3}.$$

The curves $\Gamma^+_{\eta(t)}$ and $\Gamma^-_{\eta(t)}$ are parametrized by

$$\Phi^{\pm}(t,x)=(x,\eta(t,x)\pm e),\ \ (t,x)\in(0,T)\times(0,\ell_s), \tag{3.6.13}$$

while the curve $\Gamma^\ell_{\eta(t)}$ is parametrized by

$$\Phi^{\ell_s}(t,\lambda)=(\ell_s,(1-\lambda)(-e+\eta(t,\ell_s))+\lambda(e+\eta(t,\ell_s))),\ \ (t,\lambda)\in(0,T)\times(0,1). \tag{3.6.14}$$

Then,

$$\begin{aligned}J_1&=-\int_0^{\ell_s}\sigma(\mathbf{u}(t,\Phi^+(t,x)),p(t,\Phi^+(t,x)))\mathbf{n}^+_{\eta(t)}\cdot\mathbf{u}(t,\Phi^+(t,x))|\Phi^+_x(t,x)|\,dx\\&=-\int_0^{\ell_s}\sigma(\mathbf{u}(t,\Phi^+(t,x)),p(t,\Phi^+(t,x)))\mathbf{n}^+_{\eta(t)}\eta_t\vec{\mathbf{e}}_2\sqrt{1+\eta_x^2}\,dx,\end{aligned} \tag{3.6.15}$$

and

$$\begin{aligned}J_2&=-\int_0^{\ell_s}\sigma(\mathbf{u}(t,\Phi^-(t,x)),p(t,\Phi^-(t,x)))\mathbf{n}^-_{\eta(t)}\cdot\mathbf{u}(t,\Phi^-(t,x))|\Phi^-_x(t,x)|\,dx\\&=-\int_0^{\ell_s}\sigma(\mathbf{u}(t,\Phi^-(t,x)),p(t,\Phi^-(t,x)))\mathbf{n}^-_{\eta(t)}\eta_t\vec{\mathbf{e}}_2\sqrt{1+\eta_x^2}\,dx.\end{aligned} \tag{3.6.16}$$

On the other hand, since the kinematic condition is given by

$$\mathbf{u}(t,\Phi^{\ell_s}(t,\lambda))=\Phi^{\ell_s}_t(t,x),\ \ (t,\lambda)\in(0,T)\times(0,1),$$

which is equivalent to

$$\mathbf{u}(t,\Phi^{\ell_s}(t,\lambda))=\eta_t(t,\ell_s)\vec{\mathbf{e}}_2,\ \ (t,\lambda)\in(0,T)\times(0,1), \tag{3.6.17}$$

we have that

$$\begin{aligned}
J_3 &= -\int_0^1 \sigma(\mathbf{u}(t,\Phi^{\ell_s}(t,\lambda)), p(t,\Phi^{\ell_s}(t,\lambda)))\mathbf{n}^{\ell}_{\eta(t)} \cdot \mathbf{u}(t,\Phi^{\ell_s}(t,\lambda))|\Phi^{\ell_s}_{\lambda}(t,\lambda)|\, d\lambda \\
&= -\int_0^1 \sigma(\mathbf{u}(t,\Phi^{\ell_s}(t,\lambda)), p(t,\Phi^{\ell_s}(t,\lambda)))\mathbf{n}^{\ell}_{\eta(t)}\eta_t(t,\ell_s)\vec{\mathbf{e}}_2 2e\, d\lambda \\
&= -2e\eta_t(t,\ell_s)\int_0^1 \sigma_{21}(\mathbf{u}(t,\Phi^{\ell_s}(t,\lambda)), p(t,\Phi^{\ell_s}(t,\lambda)))\, d\lambda \\
&= -2e\eta_t(t,\ell_s)\int_0^1 u_{2,x}(t,\Phi^{\ell_s}(t,\lambda))\, d\lambda.
\end{aligned} \tag{3.6.18}$$

Thus, the relations (3.6.15) and (3.6.16) suggest us to define H as

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2. \tag{3.6.19}$$

Discussion about the term J_3

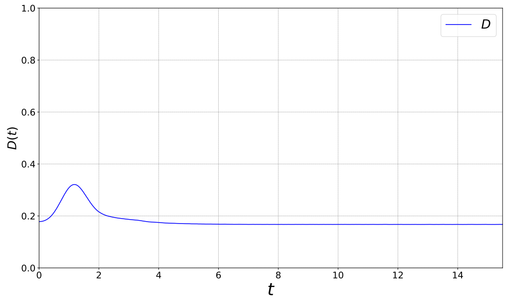
As we can notice, the term J_3 given in (3.6.18) is not necessarily zero, since neither $\eta_t(t,\ell_s)$ nor $\displaystyle\int_{\Gamma^{\ell}_{\eta(t)}} u_{2,x}$ is necessarily zero. One possible way to compensate this term in the energy identity would be to include it in one of the bondary conditions at the right end $x=\ell_s$ of the beam. This would amount to treating a structure equation with a non-standard condition. However, as a first step toward addressing such a problem in the future, we decide to consider homogeneous conditions at the right end of the beam. In this context, we present numerical results showing that the term J_3 can be neglected after comparison with the viscous dissipation term.

To assess the order of magnitude of the term $J_3 = J_3(t)$ in comparison with respect to the viscosity dissipation $D = 2\nu\displaystyle\int_{\Omega_{\eta(t)}}|\varepsilon(\mathbf{u})|^2$, we present three numerical simulations. Given a fixed Reynolds number $Re = 200$, damping coefficient $\gamma = 10^{-6}$ and the perturbation amplitude $\beta = 1.5$ of the inflow condition, we consider three different values of the rigidity coefficient α: $1, 10^{-1}$ and 10^{-2}.

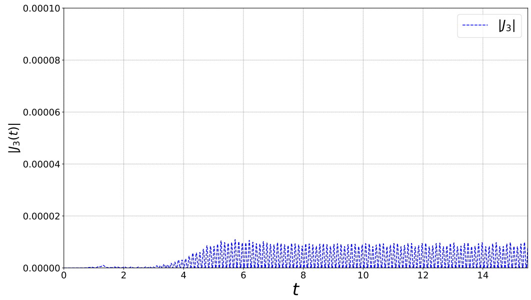
Figures 3.6, 3.7 and 3.8 show the evolution the viscosity dissipation $D = D(t)$ and $|J_3| = |J_3(t)|$ when the rigidity coefficient α is equal to 1, 10^{-1} and 10^{-2}, respectively. From these figures we deduce the following:

- When $\alpha = 1$, $D \approx 10^{-1}$, while $|J_3| \approx 10^{-5}$.
- When $\alpha = 10^{-1}$, $D \approx 10^{-1}$, while $|J_3| \approx 10^{-4}$.
- When $\alpha = 10^{-2}$, $D \approx 10^{-1}$, while $|J_3| \approx 10^{-4}$.

In all three cases analyzed, we observe that the viscous dissipation D is at least three orders of magnitude larger than J_3. Thus, it seems reasonable to neglect the contribution of J_3.

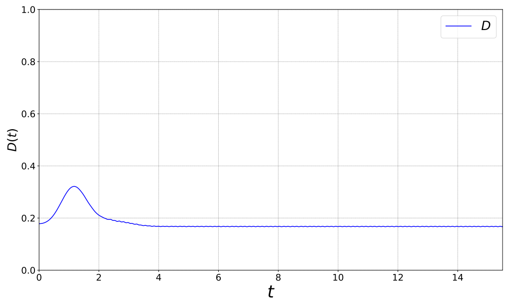


(a) $D(t)$ vs t

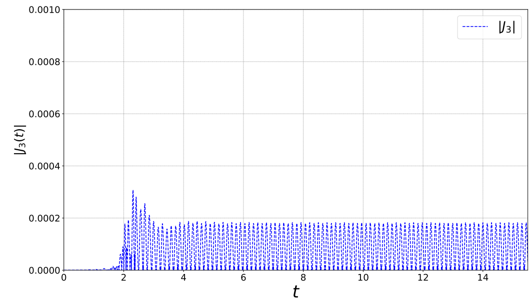


(b) $|J_3(t)|$ vs t

Figure 3.6 – Comparison between the viscosity dissipation $D = D(t)$ and $J_3 = J_3(t)$, when $\alpha = 1$.

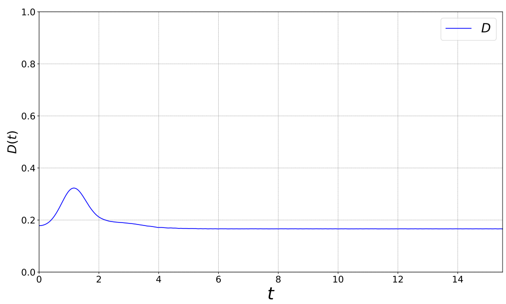


(a) $D(t)$ vs t

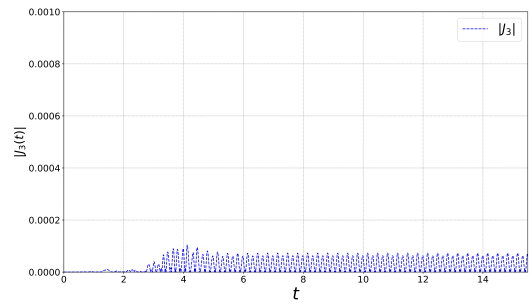


(b) $|J_3(t)|$ vs t

Figure 3.7 – Comparison between the viscosity dissipation $D = D(t)$ and $J_3 = J_3(t)$, when $\alpha = 10^{-1}$.



(a) $D(t)$ vs t



(b) $|J_3(t)|$ vs t

Figure 3.8 – Comparison between the viscosity dissipation $D = D(t)$ and $J_3 = J_3(t)$, when $\alpha = 10^{-2}$.

Appendix B: Change of variables

B.1 General change of variable formula

In this appendix we present the formulas used to obtain the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{F}_{\mathrm{div}}$ and $\widehat{F}_s$.

We set

$$\mathbf{u}(t,x) = \widehat{\mathbf{u}}(t, Y(t,x)) \text{ and } p(t,x) = \widehat{p}(t, Y(t,x)).$$

Then,

$$\begin{aligned}
&\partial_t u_i = \partial_t \widehat{u}_i + \sum_j \frac{\partial \widehat{u}_i}{\partial z_j}\frac{\partial Y_j}{\partial t}, \quad ((\mathbf{u}\cdot\nabla)\mathbf{u})_i = \sum_{j,k} \widehat{u}_j \frac{\partial \widehat{u}_i}{\partial z_k}\frac{\partial Y_k}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_j} = \sum_k \frac{\partial \widehat{u}_i}{\partial z_k}\frac{\partial Y_k}{\partial x_j},\\
&\frac{\partial^2 u_i}{\partial x_j \partial x_m} = \sum_{k,l} \frac{\partial^2 \widehat{u}_i}{\partial z_k \partial z_l}\frac{\partial Y_l}{\partial x_m}\frac{\partial Y_k}{\partial x_j} + \sum_k \frac{\partial \widehat{u}_i}{\partial z_k}\frac{\partial^2 Y_k}{\partial x_j \partial x_m}, \quad \frac{\partial p}{\partial x_i} = \sum_k \frac{\partial \widehat{p}}{\partial z_k}\frac{\partial Y_k}{\partial x_i},\\
&\operatorname{div}\mathbf{u} = \sum_{k,j} \frac{\partial \widehat{u}_j}{\partial z_k}\frac{\partial Y_k}{\partial x_j}.
\end{aligned} \tag{3.6.20}$$

B.2 Nonlinear terms coming from the geometric transformation (3.2.9)

In this section, we present the explicit formulas of the terms $\widehat{\mathbf{F}}_f$, $\widehat{F}_{\mathrm{div}}$ and $\widehat{F}_s$ that we obtain for the geometric transformation given in (3.2.9).

After some computations, we obtain:

$$\begin{aligned}
\bullet\, \widehat{\mathbf{F}}_{f,i}(\widehat{\mathbf{u}}, \widehat{p}, \eta) = &-\left(\widehat{u}_1 \frac{\partial \widehat{u}_i}{\partial z_1} - \frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_i}{\partial z_2}\widehat{u}_1 + \frac{\ell - e}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_i}{\partial z_2}\widehat{u}_2\right)\\
&+ \frac{(\ell \mp z)\chi^{\pm}\eta_t^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_i}{\partial z_2} - 2\nu\left(\frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\right)\frac{\partial^2 \widehat{u}_i}{\partial z_1 \partial z_2}\\
&+ \nu\left(\frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\right)^2 \frac{\partial^2 \widehat{u}_i}{\partial z_2 z_1} - \nu\left(\frac{2(\ell - e) + \eta_\chi^{\pm}}{(\ell - e + \eta_\chi^{\pm})^2}\right)\eta_\chi^{\pm}\frac{\partial^2 \widehat{u}_i}{\partial z_2^2}\\
&- \nu\frac{\partial}{\partial z_1}\left(\frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\right)\frac{\partial \widehat{u}_i}{\partial z_2} + \nu\frac{\partial}{\partial z_2}\left(\frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\right)\frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_i}{\partial z_2}\\
&+ \nu(\ell - e)^2 \frac{1}{\ell - e + \eta_\chi^{\pm}}\frac{\partial}{\partial z_2}\left(\frac{1}{\ell - e + \eta_\chi^{\pm}}\right)\frac{\partial \widehat{u}_i}{\partial z_2}\\
&- \nu\frac{\partial}{\partial z_i}\left(-\frac{\eta_\chi^{\pm}}{\ell - e}\frac{\partial \widehat{u}_1}{\partial z_1} + \frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e}\frac{\partial \widehat{u}_1}{\partial z_2}\right) + ((I_{\mathbb{R}^2} - J^\top)\nabla q)_i,
\end{aligned} \tag{3.6.21}$$

where,

$$(I_{\mathbb{R}^2} - J^\top)\nabla q = \frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial q}{\partial z_2}\vec{\mathbf{e}}_1 + \frac{\eta_\chi^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial q}{\partial z_2}\vec{\mathbf{e}}_2.$$

$$\bullet\, \widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}}, \eta) = -\frac{\eta_\chi^{\pm}}{\ell - e}\widehat{u}_1\vec{\mathbf{e}}_1 + \frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e}\widehat{u}_1\vec{\mathbf{e}}_2. \tag{3.6.22}$$

$$\bullet \widehat{F}_s(\widehat{\mathbf{u}}, \eta) = -\nu\gamma_s^{+}\left(\frac{\ell - e}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_1}{\partial z_2} + \frac{\partial \widehat{u}_2}{\partial z_1} - \frac{(\ell \mp z)\chi^{\pm}\partial_{z_1}\eta^{\pm}}{\ell - e + \eta_\chi^{\pm}}\frac{\partial \widehat{u}_2}{\partial z_2}\right) \tag{3.6.23}$$

$$+\nu\gamma_s^-\left(\frac{\ell-e}{\ell-e+\eta_\chi^\pm}\frac{\partial\widehat{u}_1}{\partial z_2}+\frac{\partial\widehat{u}_2}{\partial z_1}-\frac{(\ell\mp z)\chi^\pm\partial_{z_1}\eta^\pm}{\ell-e+\eta_\chi^\pm}\frac{\partial\widehat{u}_2}{\partial z_2}\right) \tag{3.6.24}$$

$$+2\nu\frac{\ell-e}{\ell-e+\eta_\chi^\pm}\gamma_s^+\left(\frac{\partial\widehat{u}_2}{\partial z_2}\right)-2\nu\frac{\ell-e}{\ell-e+\eta_\chi^\pm}\gamma_s^-\left(\frac{\partial\widehat{u}_2}{\partial z_2}\right). \tag{3.6.25}$$

Appendix C: Auxiliar results

C.1 Study of Stokes system with irregular divergence data (case 1)

Let $\alpha\in(0,\alpha^*)$. Here, we will consider the decomposition of the domain Ω as the one given in Figure 3.4. In this section, we shall study the the Stokes system

$$\begin{cases}-\operatorname{div}\sigma(\mathbf{w},p)=0 \ \text{ in } \Omega, \ \ \operatorname{div}\mathbf{w}=\operatorname{div}\mathbf{h}_{\rm div} \ \text{ in } \Omega,\\ \mathbf{w}=0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w},p)\mathbf{n}=0 \ \text{ on } \Gamma_n,\end{cases} \tag{3.6.26}$$

when $\mathbf{h}_{\rm div}$ belongs to $\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\Omega)$ satisfies $\mathbf{h}_{\rm div}|_{\mathcal{O}_\mathcal{R}}=0$. Before to introduce the definition of what we will understand by a solution of (3.6.26), let us consider the system

$$\begin{cases}-\operatorname{div}\sigma(\boldsymbol{\Phi},\psi)=\boldsymbol{\zeta} \ \text{ in } \Omega, \ \ \operatorname{div}\boldsymbol{\Phi}=0 \ \text{ in } \Omega,\\ \boldsymbol{\Phi}=0 \ \text{ on } \Gamma_d, \ \ \sigma(\boldsymbol{\Phi},\psi)\mathbf{n}=0 \ \text{ on } \Gamma_n.\end{cases} \tag{3.6.27}$$

Definition 3.C.1. *Let $\alpha\in(0,\alpha^*)$. Let us assume that $\mathbf{h}_{\rm div}\in\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\Omega)$ with $\mathbf{h}_{\rm div}|_{\mathcal{O}_\mathcal{R}}=0$. We say that $\mathbf{w}\in\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\Omega)$ is a solution to (3.6.26) in the sense of transposition if and only if*

$$\langle\mathbf{w},\boldsymbol{\zeta}\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha},\mathbf{H}_{\Gamma_d}^{\frac12-\alpha}}=\langle\mathbf{h}_{\rm div},\nabla\psi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha},\mathbf{H}_{\Gamma_d}^{\frac12-\alpha}}, \tag{3.6.28}$$

for all $\boldsymbol{\zeta}\in\mathbf{L}^2(\Omega)$, where $(\boldsymbol{\Phi},\psi)$ is solution of system (3.6.27).

Lemma 3.C.2. *Let $\alpha\in(0,\alpha^*)$. For all $\mathbf{h}_{\rm div}\in\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\Omega)$ satisfying $\mathbf{h}_{\rm div}|_{\mathcal{O}_\mathcal{R}}=0$, system (3.6.26) admits a unique solution $\mathbf{w}\in\mathbf{H}^{-\frac12+\alpha,0}(\Omega)$ in the sense of definition 3.C.1.*

Proof. Let $\boldsymbol{\zeta}\in\mathbf{H}_{\Gamma_d}^{\frac12-\alpha}(\Omega)$ satisfiying $\boldsymbol{\zeta}|_{\mathcal{O}_R}=0$. By using a localization argument and [Dau89, Theorem 5.5(b)], [MR10, Theorem 9.4.5] as in the proof of [BFGR, Theorem 3.2], we can deduce that $\psi\in H^{\frac12+\alpha}(\Omega)$ with $\psi|_{\mathcal{O}_L}\in H^{\frac32-\alpha}(\mathcal{O}_L)$. Then, arguing as in the proof of [BFGR, Theorem 17] we have that $\mathbf{w}|_{\mathcal{O}_L}\in\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\mathcal{O}_L)$. Moreover,

$$\|\mathbf{w}\|_{\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\mathcal{O}_L)}\le C\|\mathbf{h}_{\rm div}\|_{\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\mathcal{O}_L)}\|\nabla\psi\|_{\mathbf{H}^{\frac12-\alpha}(\mathcal{O}_L)}. \tag{3.6.29}$$

On the other hand, taking $\zeta\in\mathbf{L}^2(\Omega)$ with $\zeta|_{\mathcal{O}_L}=0$ and using a similar argument as above, we can deduce that in particular $\psi|_{\mathcal{O}_L}\in H^{\frac32-\alpha}(\mathcal{O}_L)$. Moreover,

$$\|\mathbf{w}\|_{\mathbf{L}^2(\mathcal{O}_R)}\le C\|\mathbf{h}_{\rm div}\|_{\mathbf{H}_{\Gamma_d}^{-\frac12+\alpha}(\mathcal{O}_L)}\|\nabla\psi\|_{\mathbf{H}^{\frac12-\alpha}(\mathcal{O}_L)}. \tag{3.6.30}$$

Then, the conclusion follows from (3.6.29) and (3.6.30). $\square$

C.2 Study of the Stokes system with irregular divergence data (case 2)

Let us consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{w}},\widetilde{\pi}) = 0, \ \operatorname{div}\widetilde{\mathbf{w}} = h_2 \ \text{ in } \Omega, \\ \widetilde{\mathbf{w}} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\widetilde{\mathbf{w}},\widetilde{\pi})\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{3.6.31}$$

Definition 3.6.1. *Let $\alpha \in (0,\alpha^*)$. Assume $h_2 \in \mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)$. We say that $\widetilde{\mathbf{w}} \in \mathbf{L}^2(\Omega)$ is solution to (3.6.31) if and only if,*

$$\int_\Omega \widetilde{\mathbf{w}}\cdot\mathbf{G} = -\left\langle h_2,\rho\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}_{\Gamma_d}(\Omega)} \quad \textit{for all } \mathbf{G}\in\mathbf{L}^2(\Omega), \tag{3.6.32}$$

where $(\boldsymbol{\varphi},\rho)$ is solution to

$$\begin{cases} -\operatorname{div}\sigma(\boldsymbol{\varphi},\rho) = \mathbf{G}, \ \operatorname{div}\boldsymbol{\varphi} = 0 \ \textit{ in } \Omega, \\ \boldsymbol{\varphi} = 0 \ \textit{ on } \Gamma_d, \ \ \sigma(\boldsymbol{\varphi},\rho)\mathbf{n} = 0 \ \textit{ on } \Gamma_n. \end{cases} \tag{3.6.33}$$

Lemma 3.6.1. *Assume $h_2 \in \mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)$. System (3.6.31) admits a unique solution $\widetilde{\mathbf{w}} \in \mathbf{L}^2(\Omega)$ in the sense of Definition 3.6.1.*

Proof. Given $\mathbf{G}\in\mathbf{L}^2(\Omega)$, we know from [MR10, Theorem 9.4.5] that there exists a unique couple $(\boldsymbol{\varphi},\rho)\in\mathbf{H}^{\frac{3}{2}+\alpha}_{\delta}(\Omega)\times H^1_\delta(\Omega)$ solution to (3.6.33). Then, since $H^1_\delta(\Omega)\hookrightarrow H^{\frac{1}{2}+\alpha}(\Omega)$, the right-hand side in (3.6.32) is well-defined.

Let us consider the linear functional $L:\mathbf{L}^2(\Omega)\longrightarrow\mathbb{R}$ defined by

$$L(\mathbf{G}) = -\left\langle h_2,\rho\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}_{\Gamma_d}(\Omega)}.$$

Then,

$$|L(\mathbf{G})| = \left|\left\langle h_2,\rho\right\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}_{\Gamma_d}(\Omega)}\right| \le C\|h_2\|_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)}\|\mathbf{G}\|_{\mathbf{L}^2(\Omega)}.$$

Thus, $L\in\mathcal{L}(L^2(\Omega),\mathbb{R})$. Then, the result follows from the Riesz representation Theorem. $\square$

C.3 Auxiliar Lemma

Throughout this section we will assume that X and Y are two separable Hilbert spaces. We will also suppose that X is continuously embedded into Y and that the range of X is dense in Y.

Let a and b two real numbers, finite or not, with $a<b$. We set

$$W(a,b) := \left\{f\in L^2(a,b;X) \mid \partial_t f\in L^2(a,b;Y)\right\} \tag{3.6.34}$$

endowed with the norm

$$\|f\|_{W(a,b)} := \left(\|f\|^2_{L^2(a,b;X)} + \|\partial_t f\|^2_{L^2(a,b;Y)}\right)^{1/2}.$$

Lemma 3.C.3. *Let $0<T<1$. If $f\in W(0,T)$, then $f\in C([0,T];[X,Y]_{1/2})$ and*

$$\|f\|_{L^\infty(0,T;[X,Y]_{1/2})} \le C\left(\|f(0)\|_{[X,Y]_{1/2}} + \|f\|_{L^2(0,T;X)} + \|\partial_t f\|_{L^2(0,T;Y)}\right), \tag{3.6.35}$$

where $C > 0$ is independent of T.

Proof. We split the proof in three steps.

• **Step 1.** *Intermediate estimate.*

We claim that there exists a constant C independent of T, such that for all $\widetilde{w} \in W(0,T)$ with $\widetilde{w}(0) = 0$, we have

$$\|\widetilde{w}\|_{L^\infty(0,T;[X,Y]_{1/2})} \leq C \left(\|\widetilde{w}\|_{L^2(0,T;X)} + \|\partial_t \widetilde{w}\|_{L^2(0,T;Y)} \right). \tag{3.6.36}$$

Indeed, from [LM72, Theorem 3.1, p. 19] we have that there exists $C > 0$ such that for all $w \in W(0,\infty)$

$$\|w\|_{L^\infty(0,\infty;[X,Y]_{1/2})} \leq C\Big(\|w\|_{L^2(0,\infty;X)} + \|\partial_t w\|_{L^2(0,\infty;Y)}\Big). \tag{3.6.37}$$

To conclude the estimate (3.6.36) we will proceed as follows. Let us first extend $\widetilde{w}$ to $W(-\infty,T)$ by defining $\widetilde{w}(t) = 0$ if $t < 0$. Then, for $t > 0$ we define $\widetilde{w}(T+t) = 3\widetilde{w}(T-t) - 2\widetilde{w}(T-2t)$. We see that this extension belongs to $W(0,\infty)$. Then, the estimate (3.6.36) follows from (3.6.37).

• **Step 2.** *Lifting.*

From [LM72, Theorem 3.2, p. 21 and Remark 3.3, p. 22] it follows that for $f_0 := f(0) \in [X,Y]_{1/2}$, there exists a lifting $\widetilde{f} \in W(0,1)$, such that $\widetilde{f}(0) = f_0$ and

$$\|\widetilde{f}\|_{W(0,1)} \leq C\|f_0\|_{[X,Y]_{1/2}}. \tag{3.6.38}$$

Let us set $\widetilde{w}(t) := f(t) - \widetilde{f}(t)$, $t \in [0,T]$. Then, by construction $\widetilde{w} \in W(0,T)$ and $\widetilde{w}(0) = 0$. Thus, using estimates (3.6.36) and (3.6.38), we get

$$\|f - \widetilde{f}\|_{L^\infty(0,T;[X,Y]_{1/2})} \leq C\Big(\|f(0)\|_{[X,Y]_{1/2}} + \|f\|_{L^2(0,T;X)} + \|\partial_t f\|_{L^2(0,T;Y)}\Big). \tag{3.6.39}$$

• **Step 3.** *Conclusion.*

Let us first notice that from estimate (3.6.38) and [LM72, Theorem 3.1, p. 19] we have

$$\|\widetilde{f}\|_{L^\infty(0,1;[X,Y]_{1/2})} \leq C\|f(0)\|_{[X,Y]_{1/2}}. \tag{3.6.40}$$

Let us now show the estimate (3.6.35). By using the estimates (3.6.38), (3.6.39) and (3.6.40), we obtain

$$\begin{aligned} \|f\|_{L^\infty(0,T;[X,Y]_{1/2})} &\leq \|f - \widetilde{f}\|_{L^\infty(0,T;[X,Y]_{1/2})} + \|\widetilde{f}\|_{L^\infty(0,T;[X,Y]_{1/2})} \\ &\leq C\Big(\|f(0)\|_{[X,Y]_{1/2}} + \|f\|_{L^2(0,T;X)} \\ &\qquad + \|\partial_t f\|_{L^2(0,T;Y)}\Big) + C\|f(0)\|_{[X,Y]_{1/2}} \\ &\leq C\Big(\|f(0)\|_{[X,Y]_{1/2}} + \|f\|_{L^2(0,T;X)} + \|\partial_t f\|_{L^2(0,T;Y)}\Big). \end{aligned} \tag{3.6.41}$$

This completes the proof.

□

Appendix D: Proof of estimates (3.5.17) and (3.5.18)

D.1 Proof of estimate (3.5.17)

It is sufficient to prove that

$$\|\partial_t\eta\|_{L^\infty(0,T;H^{a_0}(0,\ell_s))} \le C\Big(\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\partial_t\eta\|_{L^2(0,T;H^2(0,\ell_s))} + \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big). \tag{3.6.42}$$

Let us set $f := \partial_t\eta$. Notice that $f \in H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s))$, since $\eta \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$. Then, using Lemma 3.C.3 with $X = H^2_{\{0\}}(0,\ell_s)$ and $Y = L^2(0,\ell_s)$, we deduce (3.6.42).

D.2 Proof of estimate (3.5.18)

It is sufficient to show that

$$\|\partial_t\eta\|_{L^{q^*}(0,T;H^{\frac{3}{2}+a_0}(0,\ell_s))} \le C\Big(\|\partial_t\eta(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|\partial_t\eta\|_{L^2(0,T;H^2(0,\ell_s))} + \|\partial_t^2\eta\|_{L^2(0,T;L^2(0,\ell_s))}\Big). \tag{3.6.43}$$

We split the proof in three steps.

• **Step 1.** *Interpolation and scaling arguments.*

Let us set $f := \partial_t\eta$. Next, since $\eta \in H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s))$, $f \in H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s))$.

Let us first observe that by interpolation we have

$$H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s)) \hookrightarrow H^{1/4-a_0/2}(0,1;H^{3/2+a_0}(0,\ell_s)). \tag{3.6.44}$$

Moreover, there exists a positive constant C such that

$$\|g\|_{H^{1/4-a_0/2}(0,1;H^{3/2+a_0}(0,\ell_s))} \le C\|g\|^{3/4+a_0/2}_{L^2(0,1;H^2(0,\ell_s))}\|\partial_t g\|^{1/4-a_0/2}_{L^2(0,1;L^2(0,\ell_s))}, \tag{3.6.45}$$

for all $g \in H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$ satisfying $g(0) = 0$. We remark that the Poincaré inequality was used in the last estimate. On the other hand, thanks to the continuous embedding $H^{1/4-a_0/2}(0,1) \hookrightarrow L^{q^*}(0,1)$, with $q^* = 4/(1+2a_0)$, we have

$$H^{1/4-a_0/2}(0,1;H^{3/2+a_0}(0,\ell_s)) \hookrightarrow L^{q^*}(0,1;H^{3/2+a_0}(0,\ell_s)). \tag{3.6.46}$$

Moreover, there exists a positive constant C such that

$$\|g\|_{L^{q^*}(0,1;H^{3/2+a_0}(0,\ell_s))} \le C\|\zeta\|_{H^{1/4-a_0/2}(0,1;H^{3/2+a_0}(0,\ell_s))}, \tag{3.6.47}$$

for all $g \in H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$ satisfying $g(0) = 0$.
Then, after combining the estimates (3.6.45) and (3.6.46), we obtain

$$\|g\|_{L^{q^*}(0,1;H^{3/2+a_0}(0,\ell_s))} \le C\|g\|^{3/4+a_0/2}_{L^2(0,1;H^2(0,\ell_s))}\|\partial_t g\|^{1/4-a_0/2}_{L^2(0,1;L^2(0,\ell_s))}, \tag{3.6.48}$$

for all $g \in H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$ satisfying $g(0) = 0$. Let $\widetilde{g}(t) := g(t/T)$, $t \in (0,T)$. Then, since

$$\begin{aligned}
&\|\widetilde{g}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} = T^{1/q^*}\|g\|_{L^{q^*}(0,1;H^{3/2+a_0}(0,\ell_s))},\\
&\|\widetilde{g}\|_{L^2(0,T;H^2(0,\ell_s))} = T^{1/2}\|g\|_{L^2(0,1;H^2(0,\ell_s))},\\
&\|\partial_t\widetilde{g}\|_{L^2(0,T;L^2(0,\ell_s))} = T^{-1/2}\|\partial_t g\|_{L^2(0,1;L^2(0,\ell_s))},
\end{aligned} \tag{3.6.49}$$

from estimate (3.6.48) we deduce that

$$\|\widetilde{g}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} \leq C\|\widetilde{g}\|^{3/4+a_0/2}_{L^2(0,T;H^2(0,\ell_s))}\|\partial_t\widetilde{g}\|^{1/4-a_0/2}_{L^2(0,T;L^2(0,\ell_s))}, \tag{3.6.50}$$

for all $\widetilde{g} \in H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s))$ satisfying $\widetilde{g}(0)=0$. We highlight that the positive constant C in (3.6.50) is the same that appears in (3.6.48), which is independent of T.

• **Step 2.** *Lifting.*

Let us first provide the space $H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$ with the norm

$$\|\widetilde{f}\|_{H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))} := \left(\|\widetilde{f}\|^2_{L^2(0,1;H^2_{\{0\}}(0,\ell_s))} + \|\partial_t\widetilde{f}\|^2_{L^2(0,1;L^2(0,\ell_s))}\right)^{1/2}.$$

From [LM72, Theorem 3.2, p. 21 and Remark 3.3, p. 22] it follows that for $f(0)=\partial_t\eta(0) \in H^1_{\{0\}}(0,\ell_s)$, there exists a lifting $\widetilde{f} \in H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$, such that $\widetilde{f}(0)=f(0)$ and

$$\|\widetilde{f}\|_{H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))} \leq C\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)}. \tag{3.6.51}$$

Let us set $\widetilde{g}(t) := f(t)-\widetilde{f}(t)$, $t\in(0,T)$. Let us notice that by construction

$$\widetilde{f} \in H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s)) \text{ and } \widetilde{f}(0)=0.$$

Then, from estimate (3.6.50), we get

$$\|f-\widetilde{f}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} \leq C\|f-\widetilde{f}\|^{3/4+a_0/2}_{L^2(0,T;H^2(0,\ell_s))}\|\partial_t(f-\widetilde{f})\|^{1/2-a_0/2}_{L^2(0,T;L^2(0,\ell_s))}. \tag{3.6.52}$$

Let us estimate the right-hand side in (3.6.52). By using Young's inequality and estimate (3.6.51), we obtain

$$\begin{aligned}
&\|f-\widetilde{f}\|^{3/4+a_0/2}_{L^2(0,T;H^2(0,\ell_s))}\|\partial_t(f-\widetilde{f})\|^{1/4-a_0/2}_{L^2(0,T;L^2(0,\ell_s))}\\
&\leq \left(\|f\|^{3/4+a_0/2}_{L^2(0,T;H^2(0,\ell_s))} + \|\widetilde{f}\|^{3/4+a_0/2}_{L^2(0,T;H^2(0,\ell_s))}\right)\times\left(\|\partial_t f\|^{1/4-a_0/2}_{L^2(0,T;L^2(0,\ell_s))} + \|\partial_t\widetilde{f}\|^{1/4-a_0/2}_{L^2(0,T;L^2(0,\ell_s))}\right)\\
&\leq C\Big(2(3/4+a_0/2)\|f\|_{L^2(0,T;H^2(0,\ell_s))} + 2(1/4-a_0/2)\|\partial_t f\|_{L^2(0,T;L^2(0,\ell_s))}\\
&\qquad + 2C\,\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)}\Big)\\
&\leq C\Big(\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|f\|_{L^2(0,T;H^2(0,\ell_s))} + \|\partial_t f\|_{L^2(0,T;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{3.6.53}$$

Next, using (3.6.53) in (3.6.52), we get

$$\begin{aligned}
\|f-\widetilde{f}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} \leq C\Big(&\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|f\|_{L^2(0,T;H^2(0,\ell_s))}\\
&+ \|\partial_t f\|_{L^2(0,T;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{3.6.54}$$

• **Step 3.** *Conclusion.*

We claim that

$$\|\widetilde{f}\|_{L^{q^*}(0,1;H^{3/2+a_0}(0,\ell_s))} \leq C\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)}. \tag{3.6.55}$$

Since $\widetilde{f}$ belongs to $H^{2,1}_{\{0\}}((0,1)\times(0,\ell_s))$, we have that

$$\|\widetilde{f}\|_{L^2(0,1;H^2(0,\ell_s))} \leq C\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)}. \tag{3.6.56}$$

Thanks to [LM72, Theorem 3.1, p. 19] and estimate (3.6.51), we deduce that

$$\|\widetilde{f}\|_{L^\infty(0,1;H^1(0,\ell_s))} \leq C\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)}. \tag{3.6.57}$$

By interpolation, we get

$$\|\widetilde{f}(t,\cdot)\|_{H^s(0,\ell_s)} \leq C\|\widetilde{f}(t,\cdot)\|^{2-s}_{H^1(0,\ell_s)}\|\widetilde{f}(t,\cdot)\|^{s-1}_{H^2(0,\ell_s)} \quad \text{for a.e. } t\in(0,1), \text{ for all } s\in[1,2].$$

In particular, if $s = 3/2 + a_0$, we obtain

$$\|\widetilde{f}(t,\cdot)\|_{H^{3/2+a_0}(0,\ell_s)} \leq \|\widetilde{f}(t,\cdot)\|^{1/2-a_0}_{H^1(0,\ell_s)}\|\widetilde{f}(t,\cdot)\|^{1/2+a_0}_{H^2(0,\ell_s)}.$$

Thus,

$$\begin{aligned}\int_0^1 \|\widetilde{f}(t,\cdot)\|^{q^*}_{H^{3/2+a_0}(0,\ell_s)}\,dt &= \int_0^1 \|\widetilde{f}(t,\cdot)\|^{2/(1/2+a_0)}_{H^{3/2+a_0}(0,\ell_s)}\,dt \\ &\leq C\|\widetilde{f}\|^{(2(1-2a_0)/(1+2a_0))}_{L^\infty(0,1;H^1(0,\ell_s))}\int_0^1 \|\widetilde{f}(t,\cdot)\|^2_{H^2(0,\ell_s)}\end{aligned}$$

This proves the claim.

Let us now show the estimate (3.6.43). By using the estimates (3.6.51), (3.6.54) and (3.6.55), we get

$$\begin{aligned}\|f\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} &\leq \|f-\widetilde{f}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} + \|\widetilde{f}\|_{L^{q^*}(0,T;H^{3/2+a_0}(0,\ell_s))} \\ &\leq C\Big(\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|f\|_{L^2(0,T;H^2(0,\ell_s))} \\ &\qquad + \|\partial_t f\|_{L^2(0,T;L^2(0,\ell_s))}\Big) + C\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)} \\ &\leq C\Big(\|f(0)\|_{H^1_{\{0\}}(0,\ell_s)} + \|f\|_{L^2(0,T;H^2(0,\ell_s))} \\ &\qquad + \|\partial_t f\|_{L^2(0,T;L^2(0,\ell_s))}\Big).\end{aligned} \tag{3.6.58}$$

This completes the proof of estimate (3.6.43).

Chapter 4

Numerical simulations of the fluid-structure interaction system

Abstract of the current chapter

In this chapter, we deal with the numerical approximation of a fluid-structure interaction system. The system couples the incompressible Navier-Stokes equations in a two dimensional rectangular domain with an elastic structure governed by a damped Euler-Bernoulli beam equation. To deal with the change of the fluid domain over the time, we use a classical approach widely used in the literature, namely the Arbitrary Lagrangian-Eulerian (ALE) approach. We then describe the semi-implicit monolithic method employed, along with the presentation of the numerical results obtained from its implementation. We also present numerical experiments aimed at analyzing the spectrum of the fluid-structure operator.

Outline of the current chapter

4.1 Introduction **119**
4.2 A variational formulation of the continuous system **120**
4.3 ALE mapping and the time-marching process **122**
4.3.1 The time-marching process 123
4.3.2 Full discretization of the nonlinear system 124
4.4 Numerical simulations **125**
4.4.1 Computation of the spectrum 127
4.4.2 Solving the direct problem 135
Appendix A: Terms to be linearized **145**

4.1 Introduction

Thoughout this chapter, the notation used to describe the reference domain Ω and the physical domain $\Omega_{\eta(t)}$, as well as their boundaries, will be the same as that introduced in Subsection 3.1.1 of Chapter 3.

In this chapter, we deal with the numerical simulation of the fluid-structure interaction system

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q_\eta^T, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ in } Q_\eta^T, \\ \mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma_i^T, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma_r^T \cup \Sigma_w^T, \\ \mathbf{u} = \eta_t \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_\eta^T, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma_n^T, \\ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega, \\ \partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) + f_s \text{ in } (0,T)\times(0,\ell_s), \\ \eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ on } (0,T)\times\{0\}, \\ \partial_{x_1}^2\eta = 0 \ \text{ and } \ \partial_{x_1}^3\eta = 0 \ \text{ on } (0,T)\times\{\ell_s\}, \\ \eta(0) = 0 \ \text{ and } \ \partial_t\eta(0) = \eta_2^0 \ \text{ in } (0,\ell_s), \end{cases} \tag{4.1.1}$$

where $\mathbf{u}$ and p respresent the fluid velocity and pressure. Here, $\sigma(\mathbf{u},p)$ is the fluid stress tensor given by

$$\sigma(\mathbf{u},p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

with $\nu > 0$ denoting the fluid viscosity. The inflow boundary condition $\mathbf{g}^i = \mathbf{g}_s^i + \beta\mathbf{g}_p^i$, where $\mathbf{g}_s^i$ is time-independent, $\mathbf{g}_p^i$ is a time-dependent perturbation of $\mathbf{g}_s^i$, while β represents the perturbation amplitude. The elastic part of the structure is governed by the reference centerline curve η of the beam. The parameters $\alpha > 0$ and $\gamma > 0$ are constants relative to the structure. The damping operator B is given by

$$B = \Delta_s^2 = \partial_{x_1}^4, \quad D(B) = H^4_{\{0,\ell_s\}}(0,\ell_s).$$

The expression of the force exerted by the fluid on $\Gamma^+_{\eta(t)} \cup \Gamma^-_{\eta(t)}$ is given by

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2, \tag{4.1.2}$$

where

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x,\eta(t,x)\pm e), p(t,x,\eta(t,x)\pm e)),$$

and $\mathbf{n}^+_{\eta(t)}$ (resp. $\mathbf{n}^+_{\eta(t)}$) is the unit normal to $\Gamma^+_{\eta(t)}$ (resp. $\Gamma^-_{\eta(t)}$) exterior to $\Omega_{\eta(t)}$.

As previously mentioned in Chapter 3, one of the main difficulties in studying a fluid-structure interaction system lies in the fact that the fluid domain $\Omega_{\eta(t)}$ evolves over time. This issue arises not only at the continuous level, but also represents a major challenge in the context of numerical simulations. To address this difficulty, we employ the Arbitrary Lagrangian Eulerian (ALE) approach (see [DGH82]). In the literature, there exists a plethora of works in which this strategy has been adopted (see, for instance, [QTV00], [QF04], [TH06], [FGG07], [Ric15]). Let us briefly describe this approach.

Let $\Omega_{ref} \subset \mathbb{R}^2$ be a fixed reference domain. We consider the ALE map $\mathcal{A} : (0,\infty)\times\Omega_{ref} \to \mathbb{R}^2$ given by $\mathcal{A} = I_{\Omega_{ref}} + \eta_{ext}$, where $\eta_{ext} : (0,\infty)\times\Omega_{ref} \to \mathbb{R}^2$ denotes the fluid domain displacement. The map $\mathcal{A}$ is assumed to be invertible. We also assume that the fluid domain $\Omega_{\eta(t)}$ is parameterized as $\Omega_{\eta(t)} = \mathcal{A}(t,\Omega_{ref})$. The fluid domain velocity $\mathbf{w}$ is defined by $\mathbf{w} = \partial_t\mathcal{A}$. Thus, we can rewrite the incompressible Navier-Stokes equations in ALE formulation as follows:

$$\begin{cases} \partial_t \mathbf{u}|_{\mathcal{A}} - \operatorname{div} \sigma(\mathbf{u}, p) + ((\mathbf{u} - \mathbf{w}) \cdot \nabla)\mathbf{u} = 0 \;\text{ in } \Omega_{\eta(t)}, \\ \operatorname{div} \mathbf{u} = 0 \;\text{ in } \Omega_{\eta(t)}, \end{cases} \tag{4.1.3}$$

for all $t > 0$, where $\partial_t|_{\mathcal{A}}$ represents the ALE time derivative. In order to define the ALE mapping $\mathcal{A}$ we use the classical harmonic mesh motion (see, for instance, [QTV00], [RW10]).

Another important aspect in the numerical approximation of fluid-structure systems concerns the resolution strategies. Broadly speaking, we can consider two groups. The first one, known as the *Partitioned* strategy, consists of solving the fluid and the structure subproblems independently and coupling them through transmision conditions. This particular strategy allows the use of existing ad-hoc solvers, although the price is the loss of efficiency with respect to the second group described in the following. This second group, refered as *Monolithic* approach, is characterized by the fact that the fluid and structure subproblems are solved simultaneously. As pointed out in [Ric15], "this approach allows the use of implicit discretization techniques and strong coupled solvers for the whole system". One the main drawback of this strategy is the computational cost.

The strategy used in this chapter follows the monolithic approach presented in [Mur19], where the author presents a monolithic algorithm for solving a fluid-structure interaction coupling the incompressible Navier-Stokes equations and an incompressible neo-hookean structure. More precisely, a semi-implicit algorithm is employed, where *semi-implicit* is understood in the sense that the fluid domain is computed explicitly. A similar approach is proposed in [SM08] with the difference that, instead of using a monolithic strategy, a partitioned one is employed.

The remainder of this chapter is organized as follows. In Section 4.2, we present a variational formulation of system (4.1.1). In Section 4.3, we introduce the ALE mapping and describe the time-marching process used to solve this formulation. In Section 4.4, we present numerical experiments. In Subsection 4.4.1, we first present the spectrum of the fluide-structure operator for different values of the rigidity coefficient α, ceteris paribus. We then, in Subsection 4.4.2, provide the corresponding numerical simulations of the direct problem associated with the cases studied in the previous subsection.

4.2 A variational formulation of the continuous system

Before presenting the variational formulation associated with system (4.1.1) and its corresponding discretization, we begin by introducing the auxiliary variables $\eta_1 := \eta$ and $\eta_2 := \eta_{1,t}$. With this notation, the structure equations in (4.1.1) can be rewritten as a first-order system.

Thus, the full system takes the following form:

$$
\begin{cases}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q^T_{\eta_1} \\
\operatorname{div}\mathbf{u} = 0 \ \text{ in } Q^T_{\eta_1}, \\
\mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma^T_i, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma^T_r \cup \Sigma^T_w, \\
\mathbf{u} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma^T_{\eta_1}, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma^T_n \\
\mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega, \\
\eta_2 = \eta_{1,t} \text{ in } (0,T)\times(0,\ell_s), \\
\partial_t \eta_2 + \alpha\Delta_s^2 \eta_1 + \gamma B \eta_2 = H(\mathbf{u},p,\eta_1) + f_s \text{ in } (0,T)\times(0,\ell_s), \\
\eta_1 = 0 \ \text{ and } \ \partial_{x_1}\eta_1 = 0 \ \text{ on } (0,T)\times\{0\}, \\
\partial^2_{x_1}\eta_1 = 0 \ \text{ and } \ \partial^3_{x_1}\eta_1 = 0 \ \text{ on } (0,T)\times\{\ell_s\}, \\
\eta_1(0) = 0 \ \text{ and } \ \partial_2\eta(0) = \eta_2^0 \ \text{ in } (0,\ell_s).
\end{cases}
\tag{4.2.1}
$$

We set $\Gamma_0 = \Gamma_r \cup \Gamma_w$. In order to take into account the Dirichlet boundary conditions on Γ_d, we introduce the Lagrange multiplier $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_{s,top}, \boldsymbol{\lambda}_{s,bot}, \boldsymbol{\lambda}_{s,lat})^\top$, where:

$$
\begin{aligned}
\boldsymbol{\lambda}_i &\text{ is the multiplier associated to the boundary data } \mathbf{g}^i \text{ prescribed on } \Gamma_i, \\
\boldsymbol{\lambda}_0 &\text{ is the multiplier associated to the null boundary data prescribed on } \Gamma_0, \\
\boldsymbol{\lambda}_{s,top} &\text{ is the multiplier associated to the kinematic condition imposed on } \Gamma^+_{\eta_1(t)}, \\
\boldsymbol{\lambda}_{s,bot} &\text{ is the multiplier associated to the kinematic condition imposed on } \Gamma^-_{\eta_1(t)}, \\
\boldsymbol{\lambda}_{s,lat} &\text{ is the multiplier associated to the kinematic condition imposed on } \Gamma^\ell_{\eta_1(t)}.
\end{aligned}
\tag{4.2.2}
$$

Then, the variational formulation of system (4.2.1) reads as follows:

Find $\eta_1, \eta_2 \in L^2(0,T; H^2_{\{0\}}(0,\ell_s))$, $\mathbf{u} \in H^1(0,T;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega_{\eta_1(t)})) \cap L^2(0,T;\mathbf{H}^1(\Omega_{\eta_1(t)}))$, $p \in L^2(0,T;L^2(\Omega_{\eta_1(t)}))$ and $\boldsymbol{\lambda} \in L^2(0,T;\mathbf{H}^{-\frac{1}{2}}(\Gamma_d))$ such that

$$
\begin{cases}
\displaystyle\int_{\Omega_{\eta_1(t)}} \partial_t \mathbf{u}\cdot\boldsymbol{\phi} = a_f(\mathbf{u},\boldsymbol{\phi}) + b(\boldsymbol{\phi},p) + c(\mathbf{u},\mathbf{u},\boldsymbol{\phi}) + \int_{\Gamma_i} \boldsymbol{\lambda}_i\cdot\boldsymbol{\phi} + \int_{\Gamma_0}\boldsymbol{\lambda}_0\cdot\boldsymbol{\phi} \\
\qquad\qquad \displaystyle + \int_{\Gamma^+_{\eta_1(t)}} \boldsymbol{\lambda}_{s,top}\cdot\boldsymbol{\phi} + \int_{\Gamma^-_{\eta_1(t)}} \boldsymbol{\lambda}_{s,bot}\cdot\boldsymbol{\phi}, + \int_{\Gamma^\ell_{\eta_1(t)}} \boldsymbol{\lambda}_{s,lat}\cdot\boldsymbol{\phi}, \ \ \forall\boldsymbol{\phi}\in\mathbf{H}^1(\Omega_{\eta_1(t)}), \\
b(\mathbf{u},\psi) = 0, \ \ \forall\psi\in L^2(\Omega_{\eta_1(t)}), \\
\displaystyle\int_{\Gamma_i} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma_i}\mathbf{g}^i\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_i), \qquad \int_{\Gamma_0}\mathbf{u}\cdot\boldsymbol{\tau} = 0, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_0), \\
\displaystyle\int_{\Gamma^+_{\eta_1(t)}} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^+_{\eta_1(t)}} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^+_{\eta_1(t)}), \\
\displaystyle\int_{\Gamma^-_{\eta_1(t)}} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^-_{\eta_1(t)}} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^-_{\eta_1(t)}), \\
\displaystyle\int_{\Gamma^\ell_{\eta_1(t)}} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^\ell_{\eta_1(t)}} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^\ell_{\eta_1(t)}), \\
\displaystyle\int_0^{\ell_s} (\partial_t\eta_1)\zeta = \int_0^{\ell_s}\eta_2\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s), \\
\displaystyle\int_0^{\ell_s} (\partial_t\eta_2)\zeta = a^1_s(\eta_1,\zeta) + a^2_s(\eta_2,\zeta) - \int_{\Gamma^+_{\eta_1(t)}} \boldsymbol{\lambda}_{s,top}\cdot\vec{\mathbf{e}}_2\zeta \\
\qquad\qquad\qquad\qquad \displaystyle - \int_{\Gamma^-_{\eta_1(t)}} \boldsymbol{\lambda}_{s,bot}\cdot\vec{\mathbf{e}}_2\zeta + \int_0^{\ell_s} f_s\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),
\end{cases}
\tag{4.2.3}
$$

where the bilinear forms a_f, b, a_s^1 and a_s^2 are given by

$$\begin{aligned} a_f(\mathbf{v},\boldsymbol{\phi}) &= -2\nu \int_{\Omega_{\eta_1(t)}} \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\phi}), \quad b(\boldsymbol{\phi}, q) = \int_{\Omega_{\eta_1(t)}} (\operatorname{div} \boldsymbol{\phi})\, q, \\ a_s^1(\eta_1,\zeta) &= -\alpha \int_0^{\ell_s} \Delta\eta_1 \cdot \Delta\zeta, \quad a_s^2(\eta_2,\zeta) = -\gamma \int_0^{\ell_s} \Delta\eta_2 \cdot \Delta\zeta, \end{aligned} \tag{4.2.4}$$

while the trilinear form c is defined by

$$c(\mathbf{u},\mathbf{v},\boldsymbol{\phi}) = -\int_{\Omega_{\eta_1(t)}} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\phi}. \tag{4.2.5}$$

System (4.2.3) has to be completed with initial conditions.

Remark 11. In the variational formulation (4.2.3), we make an abuse of notation by using the L^2 inner product instead of the appropriate duality pairing.

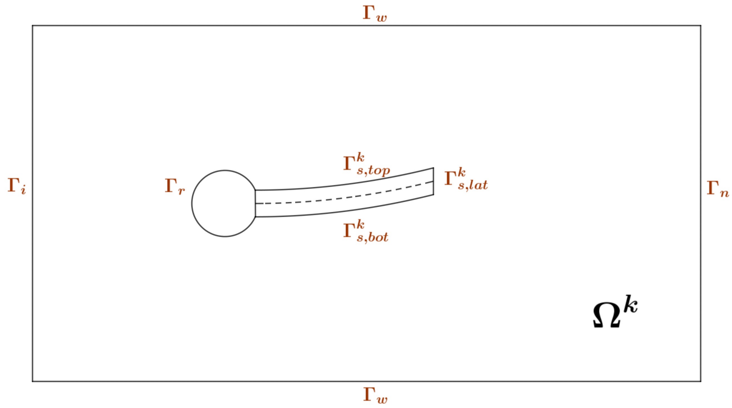


Figure 4.1 – Physical domain at time level k.

4.3 ALE mapping and the time-marching process

In this section, we introduce the ALE mapping $\mathcal{A}$ and describe the time-marching procedure used to solve problem (4.2.3).

Let us consider the ALE transformation $\mathcal{A}(t,\cdot) : \Omega_{ref} \longrightarrow \Omega_{\eta(t)}$ defined by

$$\mathcal{A}(t,\cdot) = I + \int_0^t \mathbf{w}(s,\cdot)\, ds, \tag{4.3.1}$$

where $\mathbf{w}(t,\cdot)$ is solution of the elliptic equation

$$\Delta\mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{u}|_{\Gamma_s} \text{ on } \Gamma_s, \quad \mathbf{w} = 0 \text{ on } \Gamma\setminus\Gamma_s. \tag{4.3.2}$$

Here, $\mathbf{u}|_{\Gamma_s}$ stands for the trace of the fluid velocity on Γ_s. This classical approach to defining the ALE mapping has been previously used, for instance, in [QTV00], [RW10]. This particular choice is known to be effective for small deformations of the structure (see, for instance, [Wic11]). Indeed, in Subsection 4.4.2, we present a numerical test that illustrates the deterioration of the mesh when the displacement of the structure is no longer small. To address this issue, alternative strategies should be explored. We refer [Wic11] and [HC23], for instance.

The time discretization is treated by using the classical backward Euler method. We denote by

Δt the time step and $t^k = k\Delta t$, for $k \in \mathbb{N}$ the time level k. For all $k \in \mathbb{N}$, $\Omega^k := \mathcal{A}(t^k, \Omega_{ref})$ with boundary $\Gamma^k = \Gamma_i \cup \Gamma_0 \cup \Gamma_s^k \cup \Gamma_n$, where $\Gamma_0 = \Gamma_r \cup \Gamma_w$ and $\Gamma_s^k = \Gamma_{s,top}^k \cup \Gamma_{s,bot}^k \cup \Gamma_{s,lat}^k$, see Figure 4.1. We denote by $\mathbf{u}^k$, p^k and $\boldsymbol{\lambda}^k$ the approximations of $\mathbf{u}(t^k,\cdot)$, $p(t^k,\cdot)$ and $\boldsymbol{\lambda}(t^k,\cdot)$, respectively. Here, $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}_i^k, \boldsymbol{\lambda}_0^k, \boldsymbol{\lambda}_{s,top}^k, \boldsymbol{\lambda}_{s,bot}^k, \boldsymbol{\lambda}_{s,lat}^k)^\top$ denotes the Langrange multiplier associated to the Dirichlet boundary conditions. We also denote by η_1^k and η_2^k the approximations of $\eta_1(t^k,\cdot)$ and $\eta_2(t^k,\cdot)$ defined on $(0,\ell_s)$, respectively.

Then, assuming known $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k and η_2^k, let us consider the following intermediate problem used in the description of the algorithm presented in Subsection 4.3.1:

Find $\widehat{\mathbf{u}}^{k+1} \in \mathbf{H}^1(\Omega^k)$, $\widehat{p}^{k+1} \in L^2(\Omega^k)$, $\widehat{\boldsymbol{\lambda}}^{k+1} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k \setminus \Gamma_n)$, $\eta_1^{k+1}, \eta_2^{k+1} \in H^2_{\{0\}}(0,\ell_s)$ such that

$$
\left\{
\begin{aligned}
&\int_{\Omega^k} \frac{\widehat{\mathbf{u}}^{k+1} - \mathbf{u}^k}{\Delta t} \cdot \boldsymbol{\phi} = a_f(\widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) + b(\boldsymbol{\phi}, \widehat{p}^{k+1}) + c(\widehat{\mathbf{u}}^{k+1} - \mathbf{w}^k, \widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) \\
&\qquad\qquad + \int_{\Gamma_i} \widehat{\boldsymbol{\lambda}}_i^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma_0^k} \widehat{\boldsymbol{\lambda}}_0^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma_{s,top}^k} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \boldsymbol{\phi} \\
&\qquad\qquad + \int_{\Gamma_{s,bot}^k} \widehat{\boldsymbol{\lambda}}_{s,bot}^{k+1} \cdot \boldsymbol{\phi}, + \int_{\Gamma_{s,lat}^k} \widehat{\boldsymbol{\lambda}}_{s,lat}^{k+1} \cdot \boldsymbol{\phi}, \ \ \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega^k), \\
&b(\widehat{\mathbf{u}}^{k+1}, \psi) = 0, \ \ \forall \psi \in L^2(\Omega^k), \\
&\int_{\Gamma_i} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_i} \mathbf{g}^i \cdot \boldsymbol{\tau}, \ \ \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i), \qquad \int_{\Gamma_0} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = 0, \ \ \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_0), \\
&\int_{\Gamma_{s,top}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,top}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \ \ \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,top}^k), \\
&\int_{\Gamma_{s,bot}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,bot}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \ \ \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,bot}^k), \\
&\int_{\Gamma_{s,lat}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,lat}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \ \ \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,lat}^k), \\
&\int_0^{\ell_s} \frac{\eta_1^{k+1} - \eta_1^k}{\Delta t} \zeta = \int_0^{\ell_s} \eta_2^{k+1} \zeta, \ \ \forall \zeta \in H^2_{\{0\}}(0,\ell_s), \\
&\int_0^{\ell_s} \frac{\eta_2^{k+1} - \eta_2^k}{\Delta t} \zeta = a_s^1(\eta_1^{k+1}, \zeta) + a_s^2(\eta_2^{k+1}, \zeta) \\
&\qquad\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} \\
&\qquad\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{bot}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} + \int_0^{\ell_s} f_s \zeta, \ \ \forall \zeta \in H^2_{\{0\}}(0,\ell_s).
\end{aligned}
\right.
\tag{4.3.3}
$$

where

$$
\begin{aligned}
&a_f(\mathbf{v}, \boldsymbol{\phi}) = -2\nu \int_{\Omega^k} \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\phi}), \ \ b(\boldsymbol{\phi}, q) = \int_{\Omega^k} (\operatorname{div} \boldsymbol{\phi})\, q, \ \ c(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}) = -\int_{\Omega^k} (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\phi}, \\
&a_s^1(\eta_1, \zeta) = -\alpha \int_0^{\ell_s} \Delta\eta_1 \cdot \Delta\zeta, \quad a_s^2(\eta_2, \zeta) = -\gamma \int_0^{\ell_s} \Delta\eta_2 \cdot \Delta\zeta.
\end{aligned}
$$

4.3.1 The time-marching process

In this subsection, we describe the semi-implicit algorithm used to solve the problem (4.2.3). This approach was implemented in [Mur19] for a fluid-structure interaction system coupling the incompresible Navier-Stokes equations and an incompressible neo-hookean structure.

Given the solution at the instant t^k, $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k and η_2^k at the known configuration Ω^k, the procedure to solve the time advancing scheme from k to $k+1$ level is described in Algorithm 3.

Algorithm 3: Semi-implicit algorithm

For $k \geq 1$:

1 : Solve the linear system that yields after applying the Newton algorithm in system (4.3.3) and get $\widehat{\mathbf{u}}^{k+1}$, η_1^{k+1}, η_2^{k+1}.

2 : Compute the mesh velocity $\widehat{\mathbf{w}}^{k+1} : \Omega^k \to \mathbb{R}^2$ satisfying the elliptic equation

$$\begin{cases} \Delta \widehat{\mathbf{w}}^{k+1} = 0 \ \text{ in } \Omega^k, \\ \widehat{\mathbf{w}}^{k+1} = \widehat{\mathbf{u}}^{k+1} \ \text{ on } \Gamma_s^k, \\ \widehat{\mathbf{w}}^{k+1} = 0 \ \text{ on } \Gamma \setminus \Gamma_s^k. \end{cases} \tag{4.3.4}$$

3 : Define $\mathcal{A}^k(\widehat{\mathbf{x}}) := \widehat{\mathbf{x}} + \Delta t \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}})$ and $\Omega^{k+1} := \mathcal{A}^k(\Omega^k)$.

4 : Define $\mathbf{u}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$, $p : \Omega^{k+1} \to \mathbb{R}$, $\boldsymbol{\lambda}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ and $\mathbf{w}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ by

$$\begin{aligned} &\mathbf{u}^{k+1}(x) = \widehat{\mathbf{u}}^{k+1}(\widehat{\mathbf{x}}), \ \ p^{k+1}(x) = \widehat{p}^{k+1}(\widehat{\mathbf{x}}), \ \ \boldsymbol{\lambda}^{k+1}(x) = \widehat{\boldsymbol{\lambda}}^{k+1}(\widehat{\mathbf{x}}) \\ &\text{and } \ \mathbf{w}^{k+1}(x) = \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}}), \ \ \forall x = \mathcal{A}^k(\widehat{\mathbf{x}}), \, \widehat{\mathbf{x}} \in \Omega^k. \end{aligned}$$

Remark 12. In Algorithm 3 we use the notation $\widehat{\ }$ to represent the state variables of Eulerian nature that are subsequently updated later in the algorithm. In this regard, we remark that, since the state variables η_1 and η_2 are of Lagangian nature, we do not use the notation $\widehat{\ }$.

4.3.2 Full discretization of the nonlinear system

Let $k \in \mathbb{N}$. We introduce the finite-dimensional spaces $\mathbf{V}_h \subset \mathbf{H}^1(\Omega^k)$ for the velocity, $P_h \subset L^2(\Omega^k)$ for the pressure, $\mathbf{D}_h \subset \mathbf{L}^2(\Gamma_d)$ for the multiplier and $S_h \subset H^2_{\{0\}}(0, \ell_s)$ for the structure's displacement and its velocity. We denote by $(\boldsymbol{\phi}_i)_{1 \leq i \leq N_u}$ a basis of $\mathbf{V}_h$, $(q_i)_{1 \leq i \leq N_p}$ a basis of P_h, $(\boldsymbol{\mu}_i^u)_{1 \leq i \leq N_{\lambda^u}}$ a basis of $\mathbf{D}_h$ and $(\zeta_i)_{1 \leq i \leq N_s}$ a basis of S_h. We set

$$\begin{aligned} &\mathbf{u}^k = \sum_{i=1}^{N_u} u_i^k \boldsymbol{\phi}_i, \ \ \mathbf{u}^0 = \sum_{i=1}^{N_u} u_i^0 \boldsymbol{\phi}_i, \ \ p^k = \sum_{i=1}^{N_p} p_i^k q_i, \ \ \eta_1 = \sum_{i=1}^{N_s} \eta_1^{k,i} \zeta_i, \ \ \eta_2 = \sum_{i=1}^{N_s} \eta_2^{k,i} \zeta_i, \\ &\eta_1^0 = \sum_{i=1}^{N_s} \eta_{1,0}^i \zeta_i, \ \ \eta_2^0 = \sum_{i=1}^{N_s} \eta_{2,0}^k \zeta_i, \ \ \boldsymbol{\lambda}_f^v = \sum_{i=1}^{N_\lambda^u} \lambda_{f,i}^{u,k} \boldsymbol{\mu}_i^u, \ \ \boldsymbol{\lambda}_s^u = \sum_{i=1}^{N_\lambda^u} \lambda_{s,i}^{u,k} \boldsymbol{\mu}_i^u, \\ &\mathbf{w}^k = \sum_{i=1}^{N_u} w_i^k \boldsymbol{\phi}_i, \ \ \mathbf{g}^i = \sum_{i=1}^{N_\lambda} g_i^k \boldsymbol{\mu}_i^u. \end{aligned}$$

We also introduce the corresponding coordinate vectors,

$$\begin{aligned} &\widehat{\mathbf{U}}^k = (u_1^k, \cdots, u_{N_u}^k)^\top, \ \ \mathbf{U}^0 = (u_1^0, \cdots, u_{N_u}^0)^\top, \ \ \mathbf{P}^k = (p_1^k, \cdots, p_{N_p}^k)^\top, \ \ \mathbf{N}_1^k = (\eta_1^{k,1}, \cdots, \eta_1^{k,N_s})^\top, \\ &\mathbf{N}_2^k = (\eta_2^{k,1}, \cdots, \eta_2^{k,N_s})^\top, \ \ \mathbf{N}_1^0 = (\eta_{1,0}^1, \cdots, \eta_{1,0}^{N_s})^\top, \ \ \mathbf{N}_2^0 = (\eta_{2,0}^1, \cdots, \eta_{2,0}^{N_s})^\top, \\ &\boldsymbol{\Lambda}_f^{u,k} = (\lambda_{f,1}^{u,k}, \cdots, \lambda_{f,N_\lambda^u}^{u,k})^\top, \ \ \boldsymbol{\Lambda}_s^{u,k} = (\lambda_{s,1}^{u,k}, \cdots, \lambda_{s,N_\lambda^u}^{u,k})^\top, \ \ \mathbf{W}^k = (w_1^k, \cdots, w_{N_u}^k)^\top, \\ &\boldsymbol{\Lambda}^{u,k} = (\boldsymbol{\Lambda}_f^{u,k} \ \ \boldsymbol{\Lambda}_s^{u,k})^\top, \quad \boldsymbol{\Theta}^k = (\mathbf{P}^k \ \ \boldsymbol{\Lambda}^{u,k})^\top, \ \ \mathbf{G}^{i,k} = (g_1^k, \cdots, g_{N_\lambda}^k)^\top. \end{aligned}$$

For all $1 \le i,j \le N_v$, $1 \le i \le N_p$, $1 \le \ell, m \le N_\lambda$, $1 \le r,t \le N_s$, we introduce the matrices

$$(A_{uu})_{ij} = a_f(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j),\ \ (A_{up})_{ik} = b(\boldsymbol{\phi}_i, p_k),\ \ (A_{u\lambda_f^u})_{il} = \int_{\Gamma_d^k \setminus \Gamma_s^k} \boldsymbol{\mu}_\ell^u \cdot \boldsymbol{\phi}_i,\ \ (A_{u\lambda_s^u})_{il} = \int_{\Gamma_s^k} \boldsymbol{\mu}_\ell^u \cdot \boldsymbol{\phi}_i,$$

$$(M_{uu})_{ij} = \int_{\Omega^k} \boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_i,\ \ (\widetilde{A}_{\eta_2\lambda_s^u})_{r\ell} = -\int_{\Gamma^k_{s,top} \cup \Gamma^k_{s,bot}} \boldsymbol{\mu}_\ell^u \cdot \vec{\mathbf{e}}_2 \zeta_r,\ \ (A_{\lambda_s^u \eta_2})_{\ell r} = -\int_{\Gamma_s^k} \boldsymbol{\mu}_\ell^u \cdot \vec{\mathbf{e}}_2 \zeta_r,$$

$$(M_{\lambda^u\lambda^u})_{ml} = \int_{\Gamma_d^k} \boldsymbol{\mu}_\ell^u \cdot \boldsymbol{\mu}_m^u,\ \ (A_{\eta_1\eta_2})_{rt} = (M_{\eta\eta})_{rt} = \int_0^{\ell_s} \zeta_r \cdot \zeta_t,\ \ (A_{\eta_2\eta_1})_{rt} = -\alpha \int_0^{\ell_s} \Delta\zeta_r \cdot \Delta\zeta_t,$$

$$(A_{\eta_2\eta_2})_{rt} = -\gamma \int_0^{\ell_s} \Delta\zeta_r \cdot \Delta\zeta_t,\ \ A_{\theta g} = \begin{pmatrix} 0 \\ M_{\lambda\lambda} \end{pmatrix}.$$

We also set

$$N_z = N_u + 2N_s,\ \ N_\theta = N_p + N_\lambda \ \text{ and } \ N = N_z + N_\theta.$$

Let us now introduce the following matrices:

$$M = \begin{pmatrix} M_{zz} & 0 \\ 0 & 0 \end{pmatrix} \ \text{ and } \ A = \begin{pmatrix} A_{zz} & A_{z\theta} \\ A_{\theta z} & 0 \end{pmatrix},$$

where

$$M_{zz} = \begin{pmatrix} M_{uu} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & 0 & M_{\eta\eta} \end{pmatrix},\ \ A_{zz} = \begin{pmatrix} A_{uu} & 0 & 0 \\ 0 & 0 & A_{\eta_1\eta_2} \\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2} \end{pmatrix},$$

$$A_{z\theta} = \begin{pmatrix} A_{up} & A_{u\lambda_f^u} & A_{v\lambda_s^u} \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{\eta_2\lambda_s^u} \end{pmatrix},\ \ A_{\theta z} = \begin{pmatrix} A_{up}^\top & 0 & 0 \\ A_{u\lambda_f^u}^\top & 0 & 0 \\ A_{u\lambda_s^u}^\top & 0 & A_{\lambda_s^u\eta_2} \end{pmatrix}.$$

Let us consider the vectors

$$L_w(\widehat{\mathbf{v}}) = -\left(\int_{\Omega^k} (\mathbf{w} \cdot \nabla)\widehat{\mathbf{v}} \cdot \boldsymbol{\phi}_\ell \right)_{1 \le \ell \le N_u}$$

and

$$N_f(\widehat{\mathbf{v}}) = -\left(\int_{\Omega^k} \sum_{i=1}^{N_u} \sum_{j=1}^{N_u} \widehat{v}_i \widehat{v}_j (\boldsymbol{\phi}_i \cdot \nabla)\boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_\ell \right)_{1 \le \ell \le N_u},$$

where $\widehat{\mathbf{v}} = \sum_{i=1}^{N_u} \widehat{v}_i \boldsymbol{\phi}_i$. Thus, the matrix formulation of the full discretization is given by

$$\frac{1}{\Delta t} M \begin{pmatrix} \widehat{\mathbf{U}} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \boldsymbol{\Theta} \end{pmatrix}^{k+1} = A \begin{pmatrix} \widehat{\mathbf{U}} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \boldsymbol{\Theta} \end{pmatrix}^{k+1} + \begin{pmatrix} L_{\mathbf{w}^k}(\widehat{\mathbf{U}}) \\ 0 \\ 0 \\ 0 \end{pmatrix}^{k+1} + \frac{1}{\Delta t} M \begin{pmatrix} \mathbf{U} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \boldsymbol{\Theta} \end{pmatrix}^{k} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -A_{\theta g}\mathbf{G}^i \end{pmatrix}^{k} + \begin{pmatrix} N_f(\widehat{\mathbf{U}}) \\ 0 \\ 0 \\ 0 \end{pmatrix}^{k+1}. \tag{4.3.5}$$

4.4 Numerical simulations

Let us first introduce the data used in the numerical experiments. The initial domain configuration Ω considered corresponds to the non-symmetric setting introduced in [TH06]; see Table 4.1.

Geometric parameters	**Value** [m]
Channel length (L)	2.5
Channel width (H)	0.41
Cylinder center position (C)	(0.2, 0.2)
Cylinder radius (R)	0.05
Elastic structure length (ℓ_s)	0.35
Elastic structure thickness (h)	0.02

Table 4.1 – Parameters of the domain Ω. Table extracted from [TH06].

At the inlet $\Gamma_i = \{0\} \times [0, 0.41]$ of the channel, we prescribe the Dirichlet boundary data

$$\mathbf{g}^i(t, x_1, x_2) = \mathbf{g}^i_s(x_1, x_2) + \beta \cdot \mathbf{g}^i_p(t, x_1, x_2), \tag{4.4.1}$$

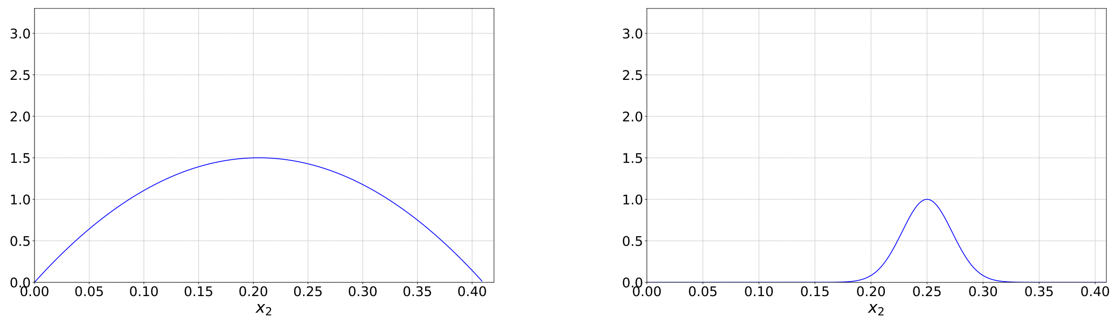
where

$$\mathbf{g}^i_s(x_1, x_2) = (g^i_{s,1}, g^i_{s,2})^\top = \left(\frac{6}{0.1681} x_2(0.41 - x_2), 0\right)^\top$$

and

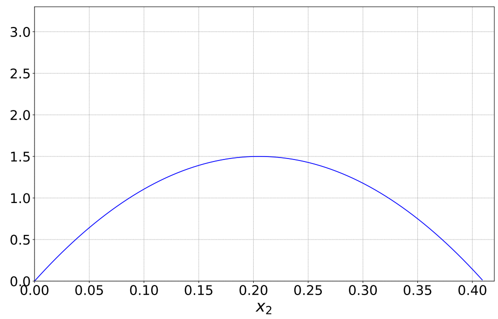
$$\mathbf{g}^i_p(t, x_1, x_2) = \begin{cases} \left(g^i_p(t) \cdot e^{-1000(x_2 - 0.25)^2}, 0\right)^\top & \text{if } t \in [0, 2], \\ (0, 0)^\top & \text{if } t > 2, \end{cases} \tag{4.4.2}$$

with $g^i_p(t) = 0.5(1 - \cos(\pi t))$ and $\beta \geq 0$ being a parameter that measures the amplitude of the perturbation $\mathbf{g}^i_p$. We highlight that the inclusion of the term $\mathbf{g}^i_p$ in the expression (4.4.1) is motivated by the study of the numerical stabilization problem treated in Chapter 6. In Figures 4.2a and 4.2b, we show the profiles of the inflow condition $g^i_{s,1}$ and the perturbation g^i_p, respectively. Figure 4.3 shows the first component of the inflow condition $\mathbf{g}^i$ at different time instants when the perturbation parameter $\beta = 1.5$. The perturbation has its greatest effect at $t = 1[s]$ (see Figure 4.3c).

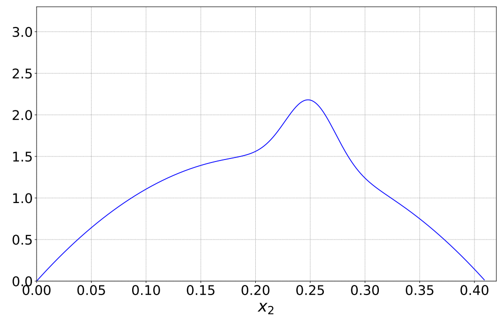


(a) Profile of the inflow condition $g^i_{s,1}$ (b) Profile perturbation g^i_p at $t = 0$.

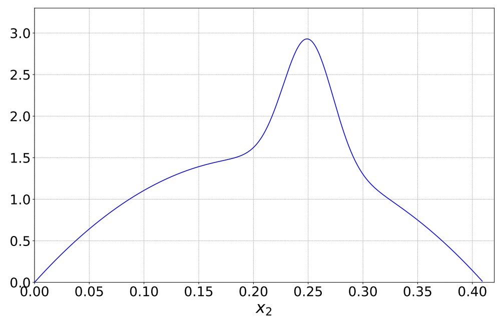
Figure 4.2 – Profile of the inflow condition $g^i_{s,1}$ and perturbation g^i_p.



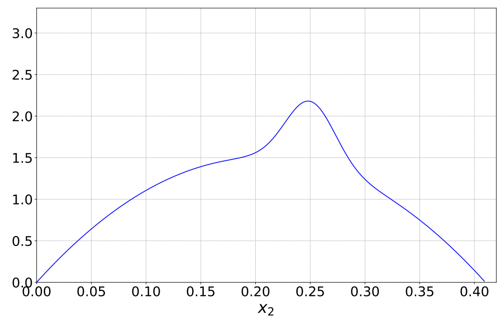
(a) $t = 0[s]$



(b) $t = 0.5[s]$



(c) $t = 1.0[s]$



(d) $t = 1.5[s]$

Figure 4.3 – Profile of the first component of the inflow data $\mathbf{g}^i$ at different time instants for the perturbation parameter $\beta = 1.5$.

The Reynolds number is defined by $Re = \dfrac{2RU_m}{\nu}$, where U_m is the mean value of $\mathbf{g}^i_{0,1}$. Thus, $Re = \frac{1}{10\nu}$ depends only on ν.

In the various numerical tests carried out in this chapter, we use a triangular mesh with 30168 cells locally refined around the structure, see Figure 4.4. For the space discretization of system (4.3.3), we choose the generalized Taylor-Hood finite elements $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$ for the velocity, the pressure and the Lagrange multiplier, respectively. The displacement and the velocity of the structure are discretized by using Hermite finite elements. The nonlinearity is treated with a Newton algorithm. The total number of degrees of freedom is equal to 394803. The numerical implementations were carried out with the open source library GetFEM++, written in C++ [RP].

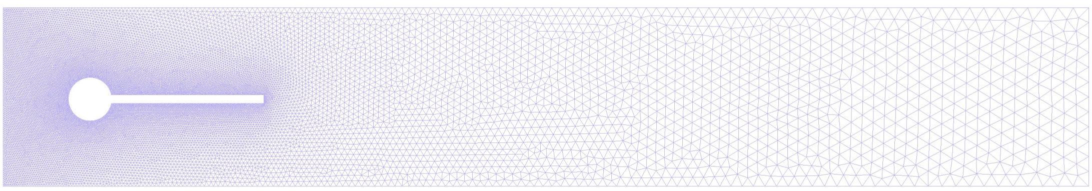


Figure 4.4 – Geometrical configuration and triangular mesh used in the numerical simulation.

4.4.1 Computation of the spectrum

An important step in the stabilization analysis to be carried out in Chapter 6 involves the spectral study of the fluid-structure operator associated with the system obtained by linearizing system (4.2.1) around a given stationary solution. This linearization processs consists of two stages: rewriting system (4.2.1) in the reference domain Ω, and subsequently linearizing it around the stationary solution. The first stage requires selecting a suitable mapping from Ω into $\Omega_{\eta(t)}$. A natural idea would be to use the same mapping introduced in Chapter 3 to study the existence of strong solutions, as done in [FNR]. However, we follow a different approach and consider a mapping like the one defined in (4.3.1).

We begin by introducing some notation before presenting the eigenvalue problem:

$$
\begin{aligned}
&\boldsymbol{\lambda}_s^d \text{ is the multiplier associated to the boundary data } \eta_2\vec{\mathbf{e}}_2 \text{ on } \Gamma_s,\\
&\boldsymbol{\lambda}_f^d \text{ is the multiplier associated to the null boundary data prescribed on } \Gamma\setminus\Gamma_s,\\
&\boldsymbol{\lambda}_s^w \text{ is the multiplier associated to the boundary data } \eta_2\vec{\mathbf{e}}_2 \text{ on } \Gamma_s,\\
&\boldsymbol{\lambda}_f^w \text{ is the multiplier associated to the null boundary data prescribed on } \Gamma\setminus\Gamma_s.
\end{aligned}
\tag{4.4.3}
$$

We thus define $\boldsymbol{\lambda}_s^d = (\boldsymbol{\lambda}_s^d, \boldsymbol{\lambda}_f^d)^\top$ and $\boldsymbol{\lambda}_s^w = (\boldsymbol{\lambda}_s^w, \boldsymbol{\lambda}_f^w)^\top$. Then, the eigenvalue problem reads as:

Find $\mu \in \mathbb{C}$, $(\mathbf{u}, p, \boldsymbol{\lambda}, \eta_1, \eta_2) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_d) \times H^2_{\{0\}}(0,\ell_s) \times H^2_{\{0\}}(0,\ell_s)$ and $(\mathbf{d}, \boldsymbol{\lambda}^d, \mathbf{w}, \boldsymbol{\lambda}^w) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_d) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_d)$ such that

$$
\left\{
\begin{aligned}
&\mu \int_\Omega \mathbf{u}\cdot\boldsymbol{\phi} = \widetilde{a}_f(\mathbf{u},\boldsymbol{\phi}) + b(\boldsymbol{\phi},p) + \int_{\Gamma_i} \boldsymbol{\lambda}_i\cdot\boldsymbol{\phi} + \int_{\Gamma_0} \boldsymbol{\lambda}_0\cdot\boldsymbol{\phi} + \int_{\Gamma^+} \boldsymbol{\lambda}_{s,top}\cdot\boldsymbol{\phi} + \int_{\Gamma^-} \boldsymbol{\lambda}_{s,bot}\cdot\boldsymbol{\phi}\\
&\qquad\qquad + \int_{\Gamma^\ell} \boldsymbol{\lambda}_{s,lat}\cdot\boldsymbol{\phi} + \int_\Omega A_1\mathbf{w}\cdot\boldsymbol{\phi} + \int_\Omega A_2\mathbf{d}\cdot\boldsymbol{\phi}, \ \ \forall\boldsymbol{\phi}\in\mathbf{H}^1(\Omega),\\
&b(\mathbf{u},\psi) = \int_\Omega A_3\mathbf{d}\psi, \ \ \forall\psi\in L^2(\Omega),\\
&\int_{\Gamma_i\cup\Gamma_0} \mathbf{u}\cdot\boldsymbol{\tau} = 0, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_i\cup\Gamma_0),\\
&\int_{\Gamma^+} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^+} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^+),\\
&\int_{\Gamma^-} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^-} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^-),\\
&\int_{\Gamma^\ell} \mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^\ell} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^\ell),\\
&\mu\int_0^{\ell_s} \eta_1\zeta = \int_0^{\ell_s} \eta_2\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
&\mu\int_0^{\ell_s} \eta_2\zeta = a_s^1(\eta_1,\zeta) + a_s^2(\eta_2,\zeta) - \int_{\Gamma^+} \boldsymbol{\lambda}_{s,top}\cdot\vec{\mathbf{e}}_2\zeta\\
&\qquad\qquad\qquad\qquad - \int_{\Gamma^-} \boldsymbol{\lambda}_{s,bot}\cdot\vec{\mathbf{e}}_2\zeta + \int_0^{\ell_s} A_4\mathbf{d}\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
&\int_\Omega \nabla\mathbf{d}:\nabla\boldsymbol{\varphi} - \int_{\Gamma_s} \boldsymbol{\lambda}_s^d\cdot\boldsymbol{\varphi} - \int_{\Gamma\setminus\Gamma_s} \boldsymbol{\lambda}_f^d\cdot\boldsymbol{\varphi} = 0, \ \ \text{for all } \boldsymbol{\varphi}\in\mathbf{H}^1(\Omega),\\
&\int_{\Gamma_s} \mathbf{d}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s} \eta_1\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0, \quad \text{for all } \boldsymbol{\tau}_s\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_s),\\
&\int_{\Gamma\setminus\Gamma_s} \mathbf{d}\cdot\boldsymbol{\tau}_f = 0, \ \ \text{for all } \boldsymbol{\tau}_f\in\mathbf{H}^{-\frac{1}{2}}(\Gamma\setminus\Gamma_s),\\
&\int_\Omega \nabla\mathbf{w}:\nabla\boldsymbol{\varphi} - \int_{\Gamma_s} \boldsymbol{\lambda}_s^w\cdot\boldsymbol{\varphi} - \int_{\Gamma\setminus\Gamma_s} \boldsymbol{\lambda}_f^w\cdot\boldsymbol{\varphi} = 0, \ \ \text{for all } \boldsymbol{\varphi}\in\mathbf{H}^1(\Omega),\\
&\int_{\Gamma_s} \mathbf{w}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0, \quad \text{for all } \boldsymbol{\tau}_s\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_s),\\
&\int_{\Gamma\setminus\Gamma_s} \mathbf{w}\cdot\boldsymbol{\tau}_f = 0, \ \ \text{for all } \boldsymbol{\tau}_f\in\mathbf{H}^{-\frac{1}{2}}(\Gamma\setminus\Gamma_s),
\end{aligned}
\right.
\tag{4.4.4}
$$

where

$$
\widetilde{a}_f(\mathbf{v},\boldsymbol{\phi}) = -2\nu\int_\Omega \varepsilon(\mathbf{v}):\varepsilon(\boldsymbol{\phi}) - \int_\Omega ((\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s)\cdot\boldsymbol{\phi},
$$

while the bilinear forms b, a_s^1 and a_s^2 are defined in (4.2.4). Here, A_1, A_2, A_3 and A_4 are linear operators, which are determined (in their discretized form) using a *GetFEM++ library* [RP] routine, after linearizing system (4.2.3) around the stationary solution $(\mathbf{u}, p, \boldsymbol{\lambda}, \eta_1, \eta_2, \mathbf{d}, \boldsymbol{\lambda}^d, \mathbf{w}, \boldsymbol{\lambda}^w) = (\mathbf{u}_s, p_s, \boldsymbol{\lambda}^s, 0, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, where the triple $(\mathbf{u}_s, p_s, \boldsymbol{\lambda}^s)$ is a stationary solution of system

$$\begin{cases} (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s - \operatorname{div} \sigma(\mathbf{u}_s, p_s) = 0, \ \ \operatorname{div} \mathbf{u}_s = 0 \ \text{ in } \Omega, \\ \mathbf{u}_s = \mathbf{g}_s^i, \ \text{ on } \Gamma_i, \ \ \mathbf{u}_s = 0 \ \text{ on } \Gamma \setminus \Gamma_i, \\ \sigma(\mathbf{u}_s, p_s)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \end{cases} \tag{4.4.5}$$

with $\boldsymbol{\lambda}^s$ representing the Lagrange multiplier associated with the Dirichlet boundary condition. We highlight that this contrast with the approach used in [FNR], where the authors carried out the linearization manually. In Appendix A, we present all the terms implemented in *GetFEM++ library*, which allow us to obtain all the linear operators $A_1, A_2, A_3, A_{41}, A_{42}, A_{43}$ and A_5 in their discretized form. For the discretization of system (4.4.4), we choose the generalized Taylor-Hood finite elements $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$ for the velocity, the pressure and the Lagrange multiplier, respectively. The displacement and the velocity of the structure are discretized by using Hermite finite elements. The displacement $\mathbf{d}$ (resp. the velocity $\mathbf{w}$) and the Lagrange multiplier $\boldsymbol{\lambda}^d$ (resp. the multiplier $\boldsymbol{\lambda}^w$) are discretized by using $\mathbb{P}_2 - \mathbb{P}_1$, respectively. The total of degree of freedom is equal to 387655. We remark that the nonlinear problem (4.4.5) is solved by using a Newton's algorithm. The magnitude of the fluid velocity $\mathbf{U}_s$ is displayed in Figure 4.5.

• **Spectral analysis with different physical parameters**. This preliminary spectral analysis of the fluid-structure operator will later allow us, in Chapter 6, to establish a comparison with the spectrum of the fluid-structure operator modified by the action of the control operator.

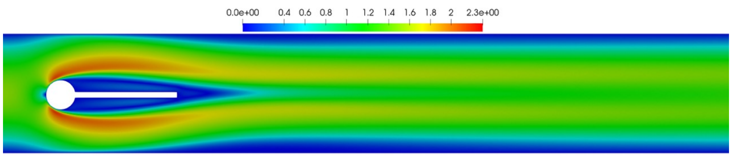


Figure 4.5 – Fluid velocity magnitude $\mathbf{U}_s$ corresponding to Reynolds number $Re = 200$.

$\boldsymbol{\alpha = 1}$. The fluid-structure spectrum displayed in Figure 4.6 shows four unstable eigenvalues, namely, $\mu_{1,2}$ and $\mu_{3,4}$ (see Table 4.2). The corresponding real parts of the eingenfunctions associated to the unstable eigenvalues are shown in Figures 4.7 (horizontal component of the fluid velocity) and 4.8 (structure's displacement).

$\boldsymbol{\mu_{1,2}}$	$\boldsymbol{\mu_{3,4}}$	$\boldsymbol{\mu_5}$	$\boldsymbol{\mu_6}$	$\boldsymbol{\mu_7}$	$\boldsymbol{\mu_{8,9}}$	$\boldsymbol{\mu_{10}}$
$1.21 \pm 25.72i$	$0.85 \pm 14.03i$	-0.23	-0.69	-0.85	$-1.43 \pm 2.64i$	-2.14

Table 4.2 – First eigenvalues of the fluid-structure system (ordered according to the real part) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 1}$, and damping coefficient $\gamma = 10^{-6}$.

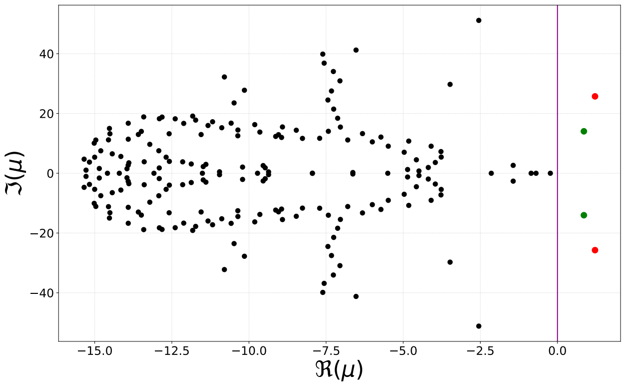
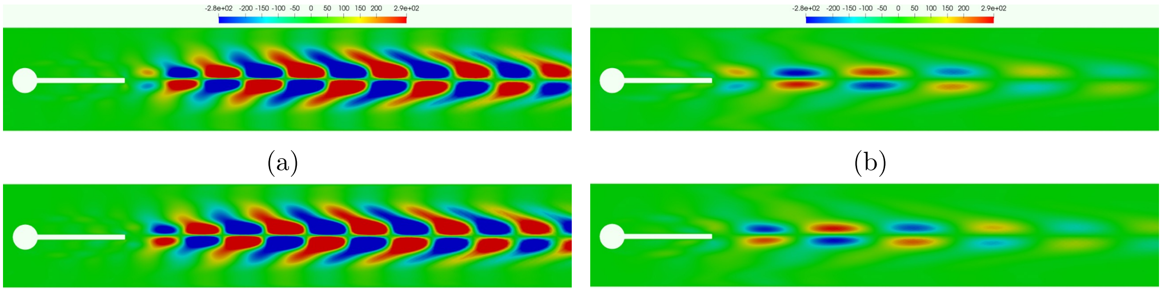


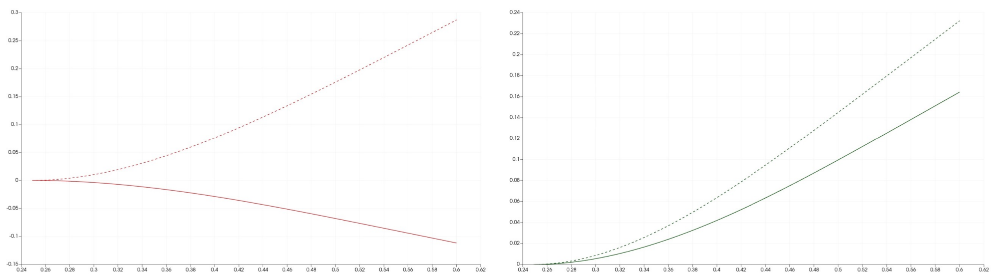
Figure 4.6 – Portion of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, and damping coefficient $\gamma = 10^{-6}$. The unstable eigenvalues are colored in red (the conjugate pair $\mu_{1,2}$) and green (the conjugate pair $\mu_{3,4}$).



(a) (b)

(c) (d)

Figure 4.7 – Real part of the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$, corresponding to the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, and damping coefficient $\gamma = 10^{-6}$. (a)-(c): Eigenfunctions associated to μ_1 and μ_2, respectively. (b)-(d): Eigenfunctions associated to μ_3 and μ_4, respectively.



(a) Eigenfunctions associated with the unstable eigenvalues $\mu_{1,2}$. The eigenfunction corresponding to μ_1 is shown in solid red, and that of μ_2 in dashed red.

(b) Eigenfunctions associated with the unstable eigenvalues $\mu_{3,4}$. The eigenfunction corresponding to μ_3 is shown in solid green, and that of μ_4 in dashed green.

Figure 4.8 – Structure's displacement of the real part of the unstable eigenmodes associated to the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$, for the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$.

<u>$\boldsymbol{\alpha} = \mathbf{10^{-1}}$</u>. In Figure 4.9 we show the fluid-structure spectrum. For this new parameter value of α and with $\gamma = 10^{-6}$ remaining constant, we observe the presence of four unstable eigenvalues, specifically $\mu_{1,2}$ and $\mu_{3,4}$ (see Table 4.3).

$\mu_{1,2}$	$\mu_{3,4}$	μ_5	μ_6	μ_7	$\mu_{8,9}$	μ_{10}
$3.05 \pm 21.61i$	$0.50 \pm 11.96i$	-0.16	-0.69	-0.79	$-1.86 \pm 27.91i$	-1.94

Table 4.3 – First eigenvalues of the fluid-structure system (ordered according to the real part) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, and damping coefficient $\gamma = 10^{-6}$.

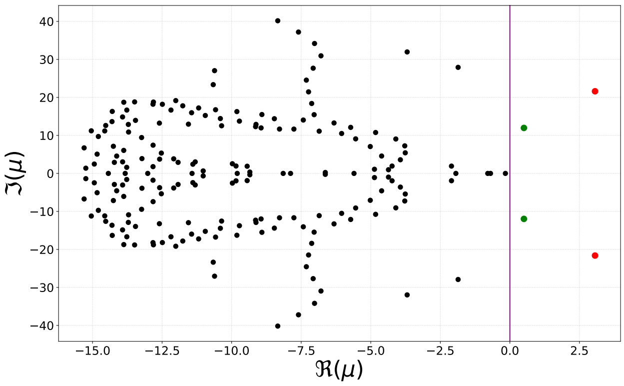
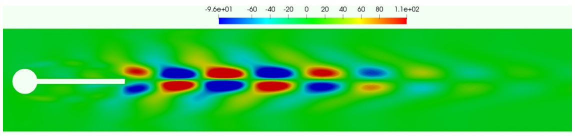
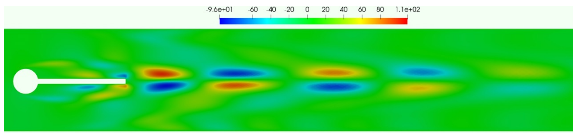


Figure 4.9 – Portion of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, and damping coefficient $\gamma = 10^{-6}$. The unstable eigenvalues are colored in red (the conjugate pair $\mu_{1,2}$) and green (the conjugate pair $\mu_{3,4}$).

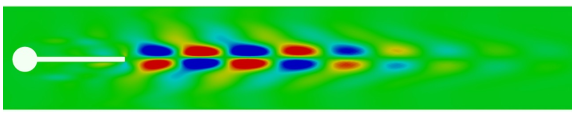
In Figures 4.10 and 4.11 we show the real part of the eigenfunctions of the structure's displacement and the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$.



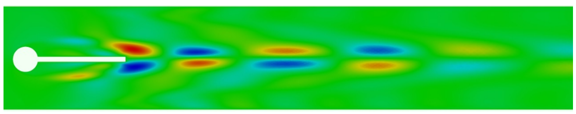
(a)



(b)

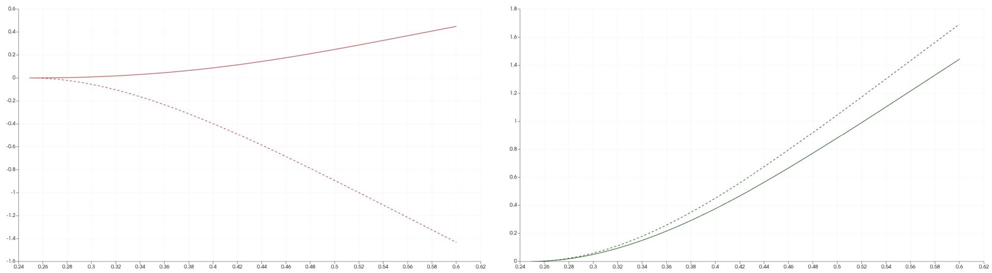


(c)



(d)

Figure 4.10 – Real part of the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, , and damping coefficient $\gamma = 10^{-6}$. (a)-(c): Eigenfunctions associated to μ_1 and μ_2, respectively. (b)-(d): Eigenfunctions associated to μ_3 and μ_4, respectively.



(a) Eigenfunctions associated with the unstable eigenvalues $\mu_{1,2}$. The eigenfunction corresponding to μ_1 is shown in solid red, and that of μ_2 in dashed red.

(b) Eigenfunctions associated with the unstable eigenvalues $\mu_{3,4}$. The eigenfunction corresponding to μ_3 is shown in solid green, and that of μ_4 in dashed green.

Figure 4.11 – Structure's displacement of the real part of the unstable eigenmodes associated to the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$, for the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$.

$\underline{\boldsymbol{\alpha = 10^{-2}}}$. In constrast to the two cases presented above, when $\alpha = 10^{-2}$, we observe from Figure 4.12 the presence of five unstable eigenvalues, namely $\mu_{1,2}$, μ_3 and $\mu_{4,5}$ (see Table 4.4), with μ_3 being a real eigenvalue.

$\boldsymbol{\mu_{1,2}}$	$\boldsymbol{\mu_3}$	$\boldsymbol{\mu_{4,5}}$	$\boldsymbol{\mu_6}$	$\boldsymbol{\mu_7}$	$\boldsymbol{\mu_{8,9}}$	$\boldsymbol{\mu_{10}}$
$2.0 \pm 19.21i$	0.38	$0.021 \pm 8.27i$	-0.48	-0.68	$-0.75 \pm 43.33i$	-1.04

Table 4.4 – First eigenvalues of the fluid-structure system (ordered according to the real part) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$.

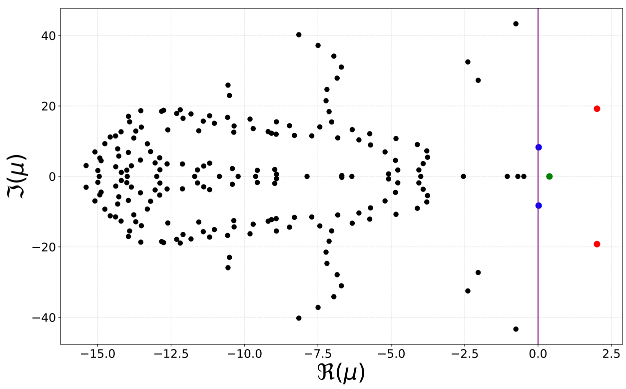
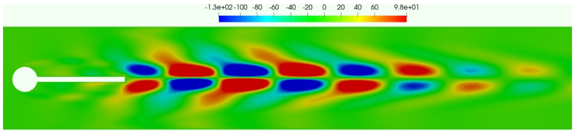
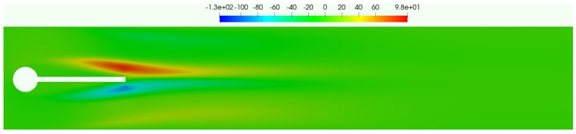


Figure 4.12 – Portion of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$. The unstable eigenvalues are colored in red (the conjugate pair $\mu_{1,2}$) and green (the real eigenvalue μ_3) and blue (the conjugate pair $\mu_{4,5}$).

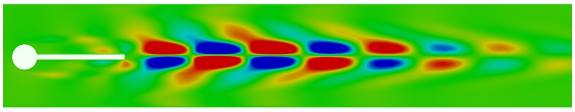
In Figures 4.13 and 4.14 we show the real part of the eigenfunctions of the structure's displacement and the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$, $\mu_{3,4}$ and μ_5.



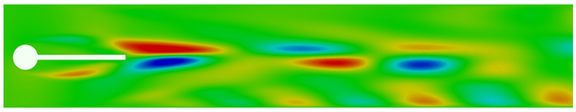
(a)



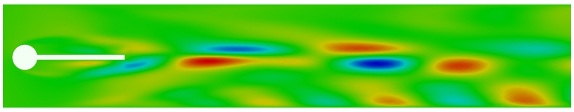
(b)



(c)

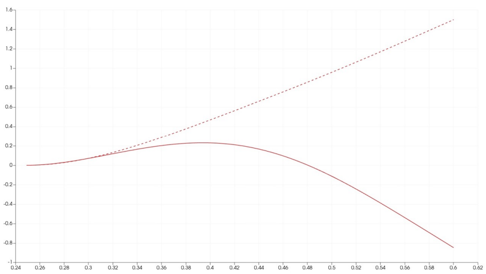


(d)

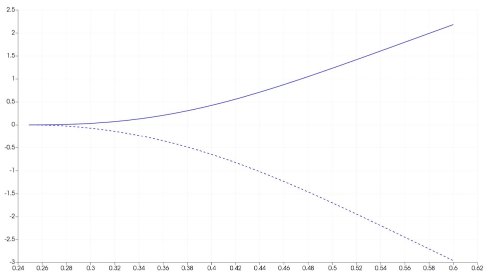


(e)

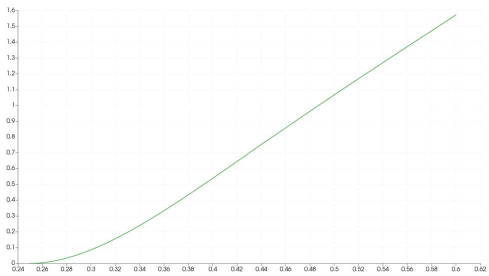
Figure 4.13 – Real part of the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$, μ_3 and $\mu_{4,5}$, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$. (a)-(c): Eigenfunctions associated to μ_1 and μ_2, respectively. (b): Eigenfunction associated to μ_3. (e)-(f): Eigenfunctions associated to μ_4 and μ_5, respectively.



(a) Eigenfunctions associated with the unstable eigenvalues $\mu_{1,2}$. The eigenfunction corresponding to μ_1 is shown in solid red, and that of μ_2 in dashed red.



(b) Eigenfunctions associated with the unstable eigenvalues $\mu_{4,5}$. The eigenfunction corresponding to μ_4 is shown in solid blue, and that of μ_5 in dashed blue.



(c) Eigenfunctions associated with the unstable eigenvalues μ_3.

Figure 4.14 – Structure's displacement of the real part of the unstable eigenmodes associated to the unstable eigenvalues $\mu_{1,2}$, μ_3 and $\mu_{4,5}$, for the rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$.

$\underline{\boldsymbol{\alpha = 10^{-3}}}$. Similarly to what was observed in the previous case when $\alpha = 10^{-2}$, we see that the fluid-structure spectrum exhibits five unstable eigenvules when $\alpha = 10^{-3}$, namely, $\mu_{1,2}$, $\mu_{3,4}$ and μ_5 (see Table 4.5).

$\boldsymbol{\mu_{1,2}}$	$\boldsymbol{\mu_3}$	$\boldsymbol{\mu_{4,5}}$	$\boldsymbol{\mu_6}$	$\boldsymbol{\mu_7}$	$\boldsymbol{\mu_{8,9}}$	$\boldsymbol{\mu_{10}}$
$2.85 \pm 18.64i$	0.23	$0.17 \pm 25.34i$	-0.43	-0.65	-1.00	$-1.49 \pm 5.45i$

Table 4.5 – First eigenvalues of the fluid-structure system (ordered according to the real part) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-3}}$, and damping coefficient $\gamma = 10^{-6}$.

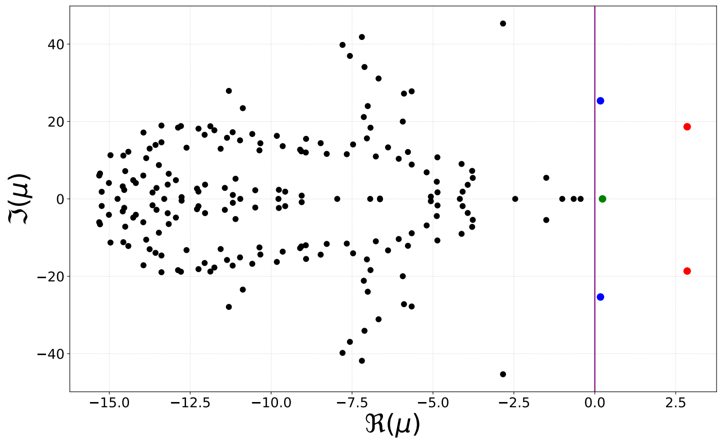
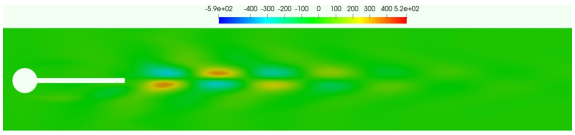
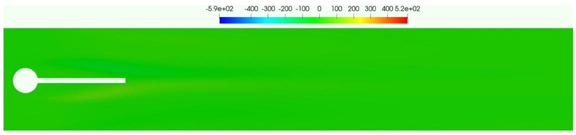


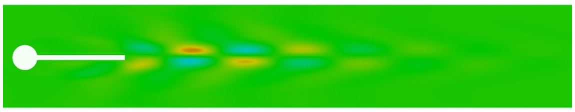
Figure 4.15 – Portion of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-3}}$, and damping coefficient $\gamma = 10^{-6}$. The unstable eigenvalues are colored in red (the conjugate pair $\mu_{1,2}$) and green (the real eigenvalue μ_3) and blue (the conjugate pair $\mu_{4,5}$).



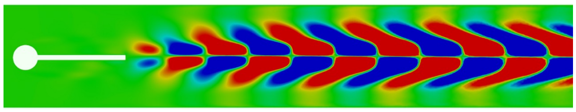
(a)



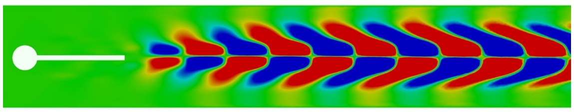
(b)



(c)

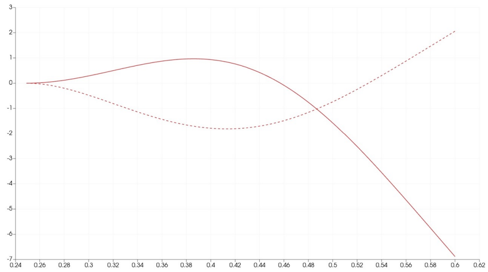


(d)

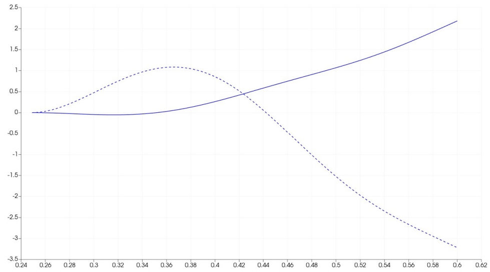


(e)

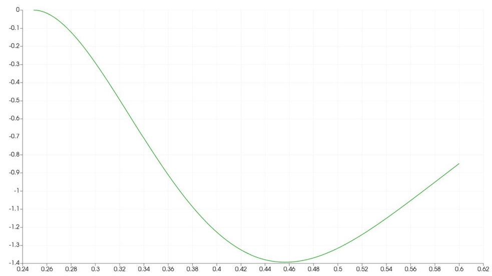
Figure 4.16 – Real part of the horizontal component of the fluid velocity associated with the unstable eigenvalues $\mu_{1,2}$ and $\mu_{3,4}$, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-3}}$, and damping coefficient $\gamma = 10^{-6}$. (a)-(c): Eigenfunctions associated to μ_1 and μ_2, respectively. (b): Eigenfunction associated to μ_3. (d)-(e): Eigenfunctions associated to μ_4 and μ_5, respectively. .



(a) Eigenfunctions associated with the unstable eigenvalues $\mu_{1,2}$. The eigenfunction corresponding to μ_1 is shown in solid red, and that of μ_2 in dashed red.



(b) Eigenfunctions associated with the unstable eigenvalues $\mu_{4,5}$. The eigenfunction corresponding to μ_4 is shown in solid blue, and that of μ_5 in dashed blue.



(c) Eigenfunctions associated with the unstable eigenvalues μ_3.

Figure 4.17 – Structure's displacement of the real part of the unstable eigenmodes associated to the unstable eigenvalues $\mu_{1,2}$, μ_3 and $\mu_{4,5}$, for the rigidity coefficient $\boldsymbol{\alpha = 10^{-3}}$.

4.4.2 Solving the direct problem

In this subsection, we present some numerical simulations of the direct problem (4.3.3) by using the time-marching Algorithm 3. In all the numerical simulations carried out in this subsection we use the following scheme for the space discretization of system (4.4.4). We choose the generalized Taylor-Hood finite elements $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$ for the velocity, the pressure and the Lagrange multiplier, respectively. The displacement and the velocity of the structure are discretized by using Hermite finite elements. The velocity $\mathbf{w}$ and the Lagrange multiplier are discretized by using $\mathbb{P}_2 - \mathbb{P}_1$, respectively. The total of degree of freedom is equal to 263759.

• **Test temporal convergence of the Algorithm 3**. To carry out this simulation, we consider the following physical parameters:

$$Re = 200, \quad \alpha = 10^{-1}, \quad \gamma = 10^{-6}.$$

We also fix the perturbation paramter $\beta = 0.1$. We denote by $(\mathbf{U}_s, P_s)$ the pair that represents the numerical approximation of the solution of system (4.4.5).

In Figure 4.18, we show the evolution of $\mathbf{L}^2-$norm of the difference $\mathbf{U} - \mathbf{U}_s$ for different time steps: $\Delta t = 10^{-3}, 5 \cdot 10^{-4}, 2.5 \cdot 10^{4}, 10^{-4}$.

Although for a given fixed mesh, as the one shown in Figure 4.4, we do not prove the convergence of Algorithm 3 as $\Delta t \longrightarrow 0$, we can infer from Figure 4.18 that, as the time step becomes smaller, we can at least qualitatively observe such convergence. This is also observed in Figure 4.19, which show the evolution of the $L^\infty-$norm of the structure's displacement for different time steps.

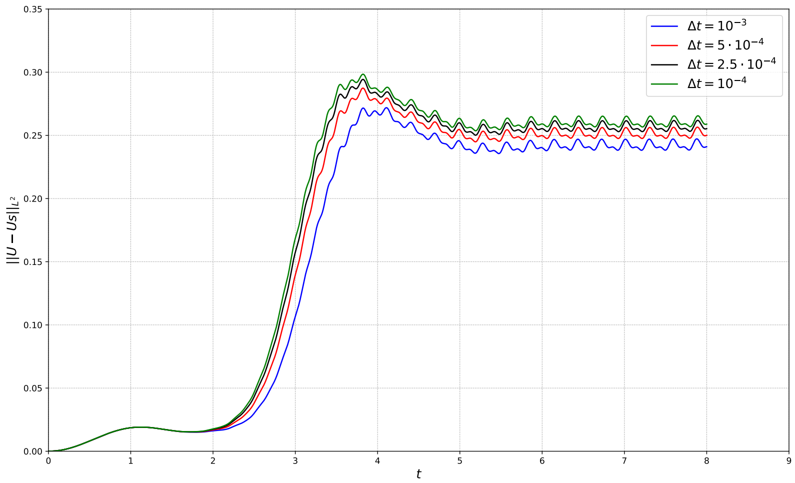


Figure 4.18 – Evolution of $\mathbf{L}^2-$norm of $\mathbf{U}-\mathbf{U}_s$ for different time steps.

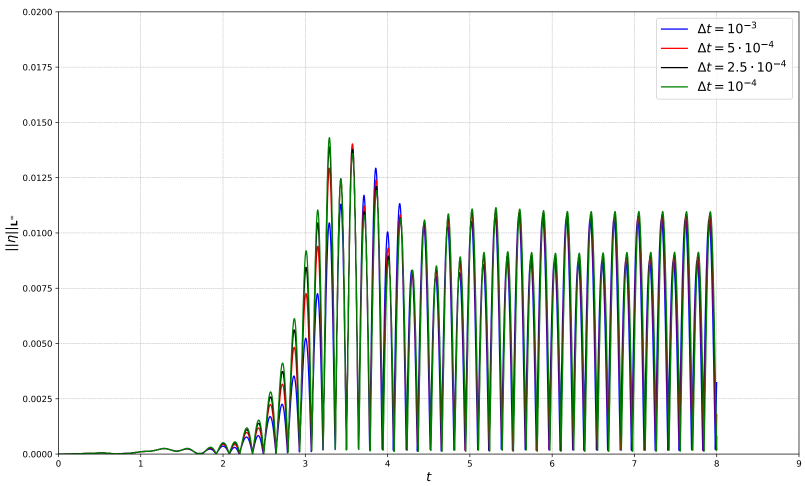
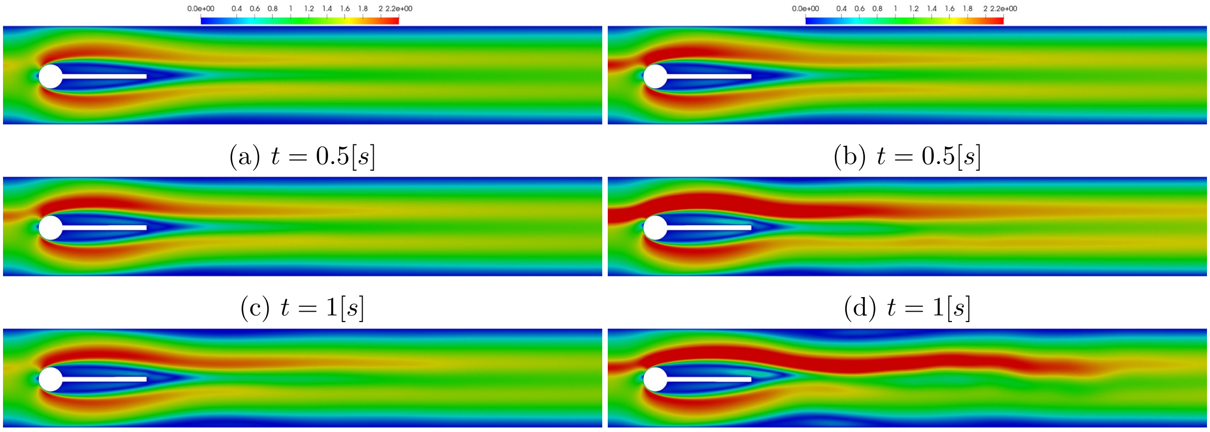


Figure 4.19 – Evolution of $L^\infty-$norm of η for different time steps.

• **Numerical simulations of the direct problem**. In this subsection, we present several numerical simulations in which we analyze the behaviour dynamics of the solution to the fluid-structure interaction problem by considering different values of the rigidity coefficient α. We also consider different levels of the perturbation β at the inflow of the channel. We emphasize that this analysis will later allow us, in Chapter 6, to evaluate the performance of the control used to stabilize the fluid-structure interaction system.

In order to facilitate the comparison, the same scale has been used in the plots showing the structure's displacement η (see Figures 4.22, 4.24, 4.26, 4.28, 4.30, 4.32).



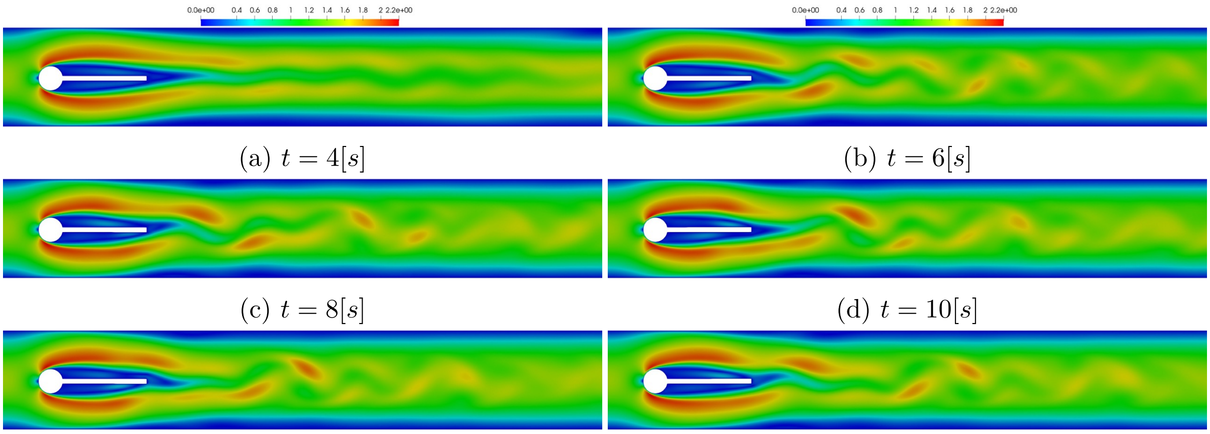
(a) $t = 0.5[s]$ (b) $t = 0.5[s]$

(c) $t = 1[s]$ (d) $t = 1[s]$

(e) $t = 1.5[s]$ (f) $t = 1.5[s]$

Figure 4.20 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 1$. In the left column (a)-(c)-(e), the velocity magnitudes are shown for perturbation parameter $\boldsymbol{\beta = 0.5}$, while in the right column (b)-(d)-(f), we display the velocity magnitude corresponding to $\boldsymbol{\beta = 1.5}$.

$\underline{\boldsymbol{\alpha = 1}}$. We fix the rigidity coefficient $\alpha = 1$. Figures 4.21 and 4.22 show snapshots of the fluid velocity magnitude and the deflection of the structure at different time instants, respectively, for $\beta = 0.5$.



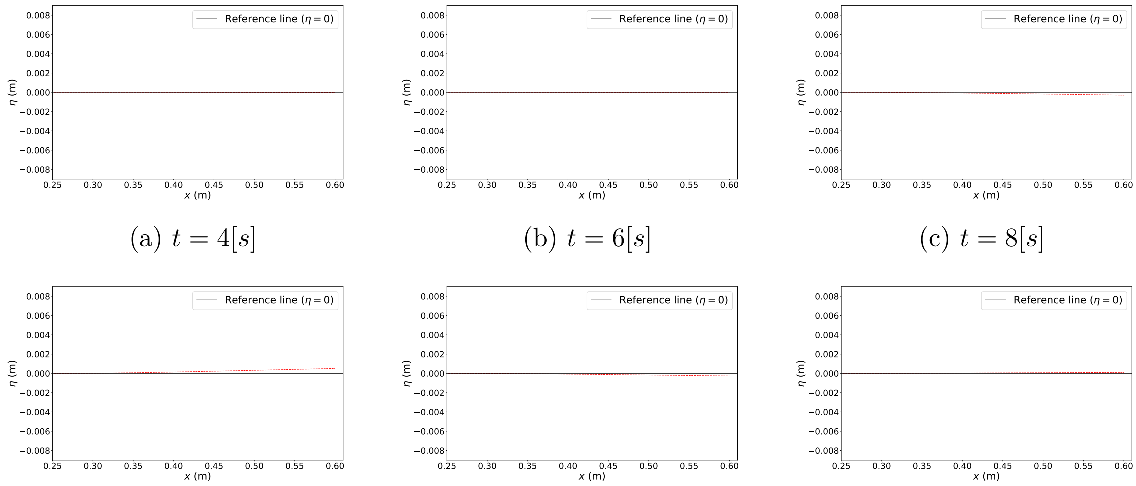
(a) $t = 4[s]$ (b) $t = 6[s]$

(c) $t = 8[s]$ (d) $t = 10[s]$

(e) $t = 12[s]$ (f) $t = 14[s]$

Figure 4.21 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 1$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.

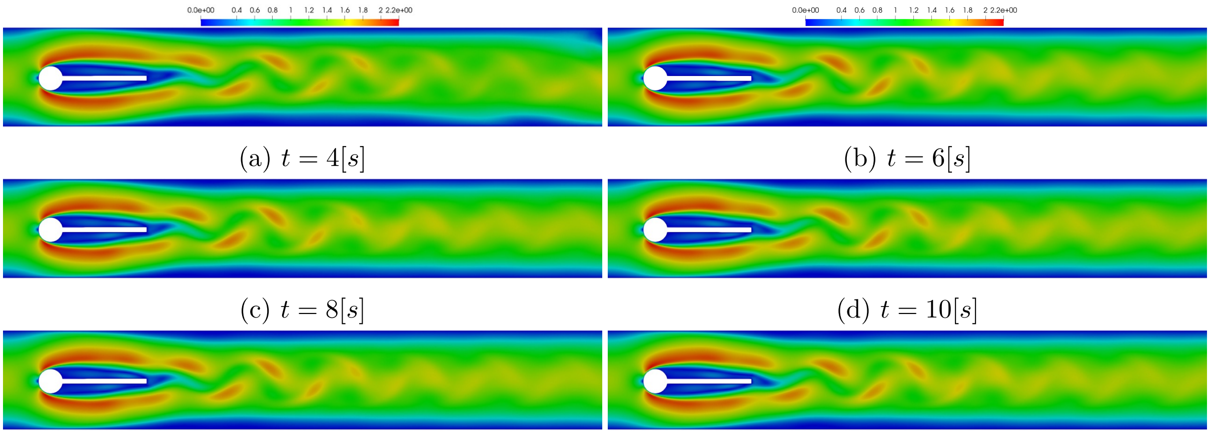
In Figures 4.23 and 4.24, we show snapshots of the fluid velocity magnitude and the deflection of the structure at different time instants, respectively, for $\beta = 1.5$.



(a) $t = 4[s]$ (b) $t = 6[s]$ (c) $t = 8[s]$

(d) $t = 10[s]$ (e) $t = 12[s]$ (f) $t = 14[s]$

Figure 4.22 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 1$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.

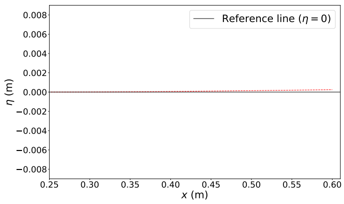


(a) $t = 4[s]$ (b) $t = 6[s]$

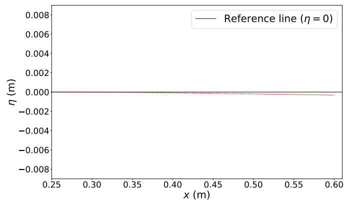
(c) $t = 8[s]$ (d) $t = 10[s]$

(e) $t = 12[s]$ (f) $t = 14[s]$

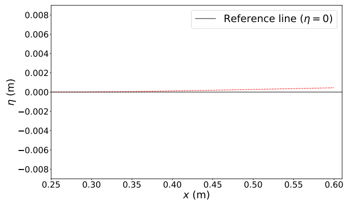
Figure 4.23 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 1$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.



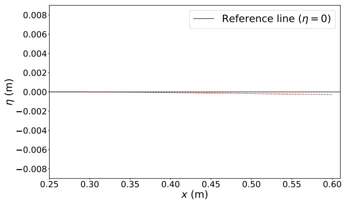
(a) $t = 4[s]$



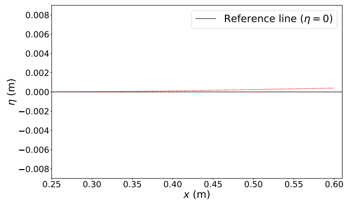
(b) $t = 6[s]$



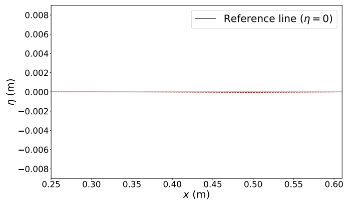
(c) $t = 8[s]$



(d) $t = 10[s]$



(e) $t = 12[s]$



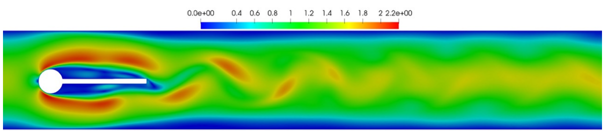
(f) $t = 14[s]$

Figure 4.24 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 1$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.

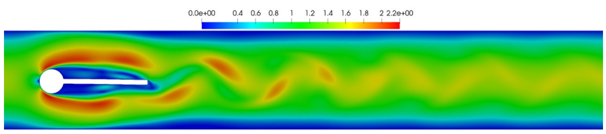
As observed in Figures 4.21 and 4.23, the dynamic of the fluid appears to be similar in both cases starting from time $t = 6[s]$, for $\beta = 0.5$ and $\beta = 1.5$. However, from Figures 4.21a and 4.25a we see that the dynamics of the fluid at $t = 4[s]$ are not the same. This is explained by the size of the perturbutation considered, as shown in Figure 4.20, which present some snapshots of the fluid velocity magnitude at $t = 0.5[s]$, $t = 1[s]$ and $t = 1.5[s]$. In this figure, we can observe how the perturbation propagates towards the region where the structure is located.

In both cases, for $\beta = 0.5$ and $\beta = 1.5$, we observe that the deflection of the structure remains relatively small, as shown in Figures 4.22 and 4.24. This can be attributed to the fact that the rigidity coefficient α is relatively "large" compared to the one used in the experiments shown below, where smaller values of this coefficient are considered.

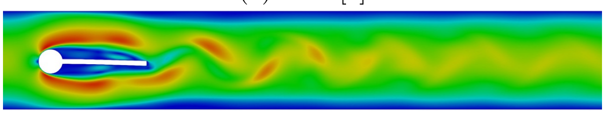
$\underline{\boldsymbol{\alpha = 10^{-1}}}$. In this experiment, we fix the rigidity coefficient $\alpha = 10^{-1}$. Figures 4.25 (resp. Figure 4.27) and 4.26 (resp. Figure 4.28) show snapshots of the fluid velocity magnitude and the deflection of the structure at different time instants, respectively, for $\beta = 0.5$ (resp. $\beta = 1.5$).



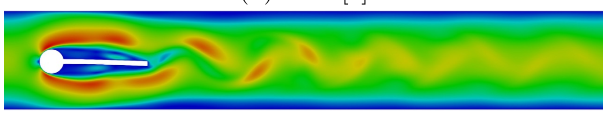
(a) $t = 4[s]$



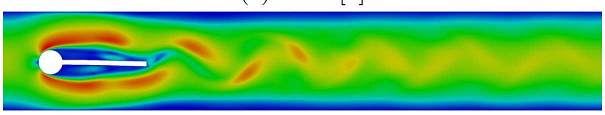
(b) $t = 6[s]$



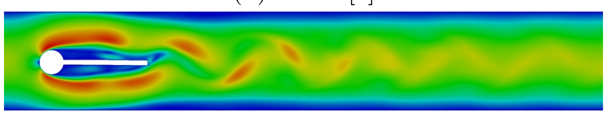
(c) $t = 8[s]$



(d) $t = 10[s]$

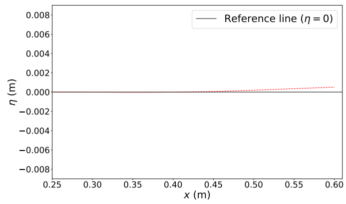


(e) $t = 12[s]$

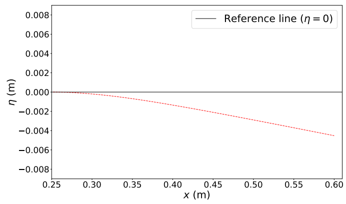


(f) $t = 14[s]$

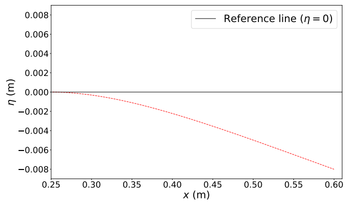
Figure 4.25 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-1}$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.



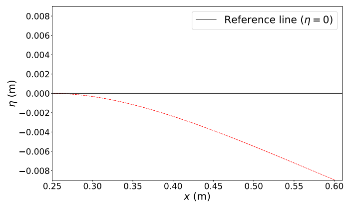
(a) $t = 4[s]$



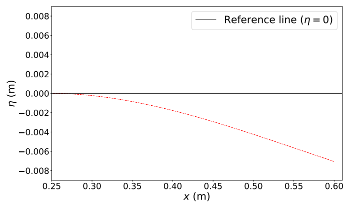
(b) $t = 6[s]$



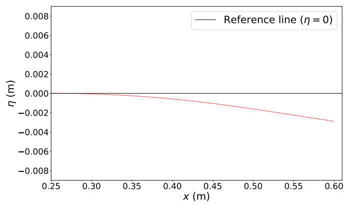
(c) $t = 8[s]$



(d) $t = 10[s]$



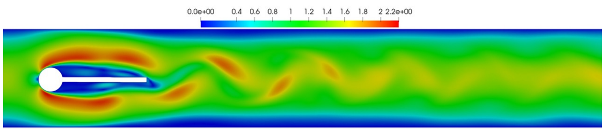
(e) $t = 12[s]$



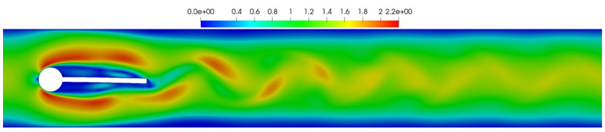
(f) $t = 14[s]$

Figure 4.26 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-1}$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.

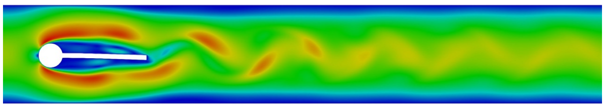
As we can observe from Figures 4.25 and 4.27, we notice that the fluid dynamics are similar, for $\beta = 0.5$ and $\beta = 1.5$. On the other hand, from Figures 4.26 and 4.28, we see that the structure's displacement at the plotted time instants is larger than that observed when $\alpha = 1$ (see Figures 4.22 and 4.24).



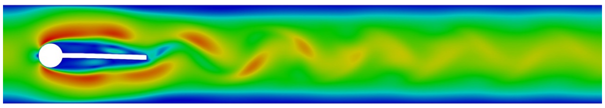
(a) $t = 4[s]$



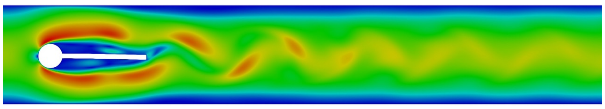
(b) $t = 6[s]$



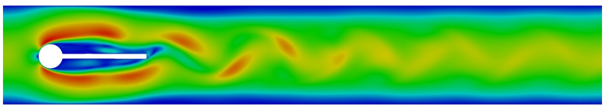
(c) $t = 8[s]$



(d) $t = 10[s]$

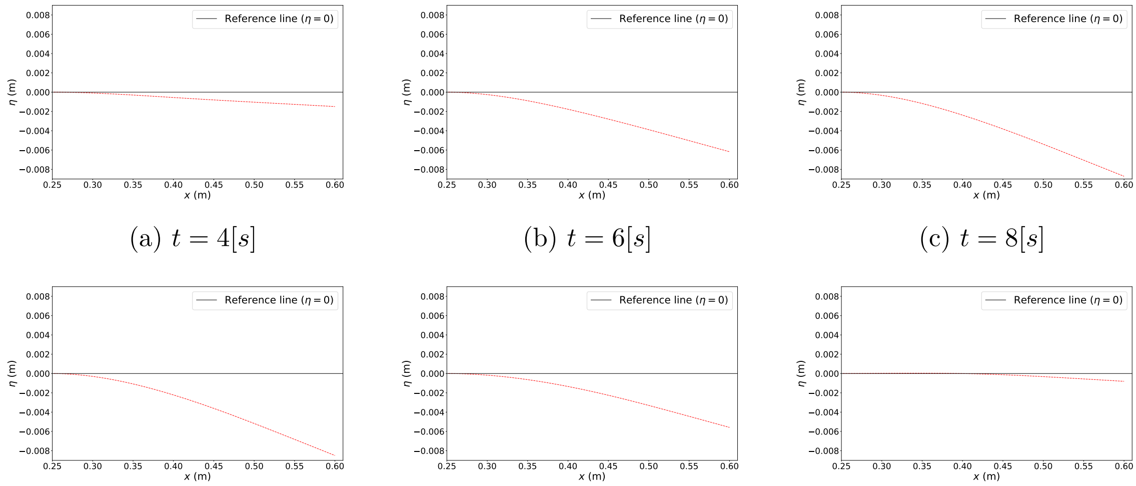


(e) $t = 12[s]$



(f) $t = 14[s]$

Figure 4.27 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-1}$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.

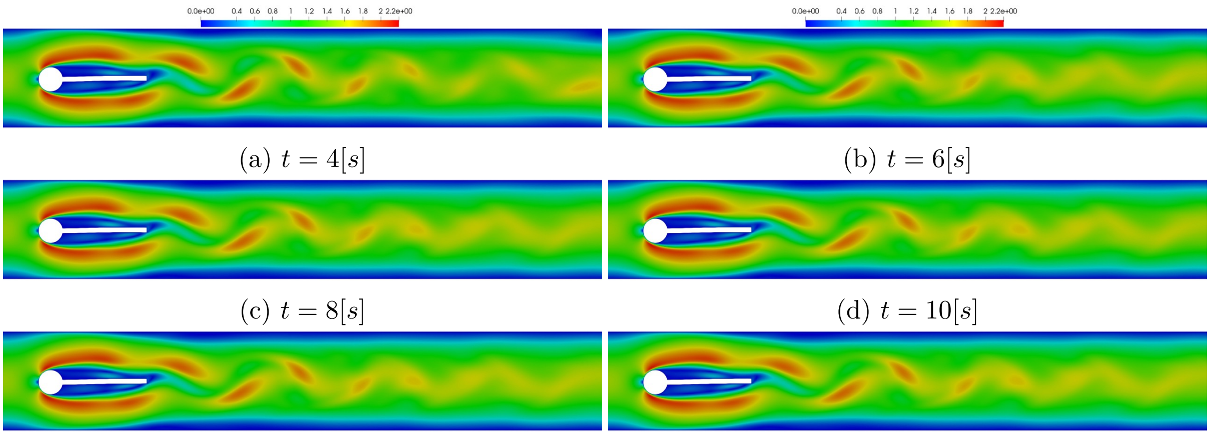


(a) $t = 4[s]$ (b) $t = 6[s]$ (c) $t = 8[s]$

(d) $t = 10[s]$ (e) $t = 12[s]$ (f) $t = 14[s]$

Figure 4.28 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-1}$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.

$\underline{\boldsymbol{\alpha = 10^{-2}}}$. In this experiment, we consider the rigidity coefficient $\alpha = 10^{-2}$. Figures 4.29 (resp. Figure 4.31) and 4.30 (resp. Figure 4.32) show snapshots of the fluid velocity magnitude and the deflection of the structure at different time instants, respectively, for $\beta = 0.5$ (resp. $\beta = 1.5$).

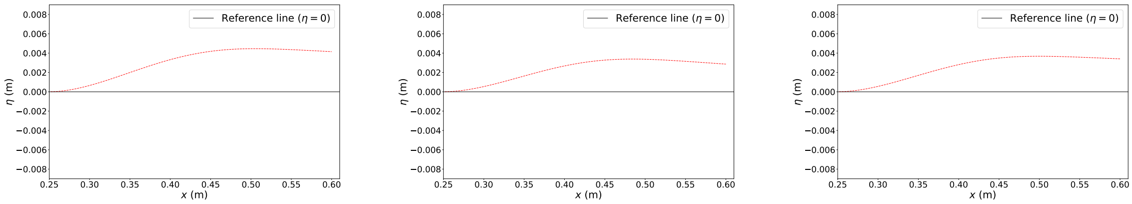


(a) $t = 4[s]$ (b) $t = 6[s]$

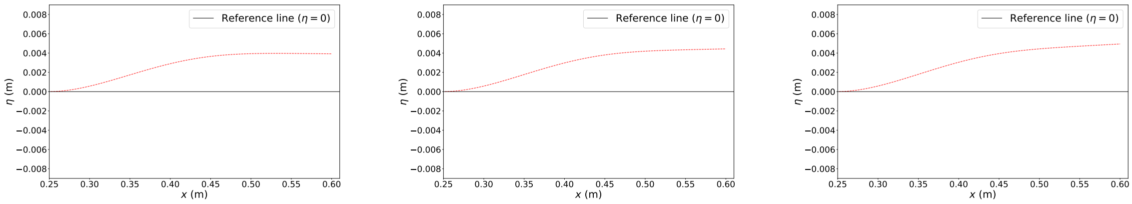
(c) $t = 8[s]$ (d) $t = 10[s]$

(e) $t = 12[s]$ (f) $t = 14[s]$

Figure 4.29 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 0.01$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.

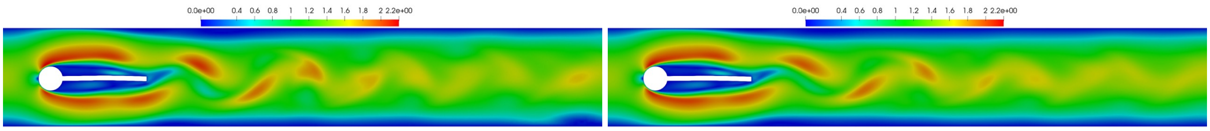


(a) $t = 4[s]$ (b) $t = 6[s]$ (c) $t = 8[s]$

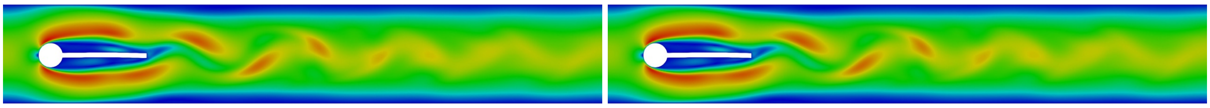


(d) $t = 10[s]$ (e) $t = 12[s]$ (f) $t = 14[s]$

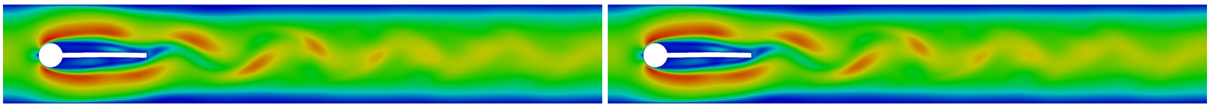
Figure 4.30 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-2}$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.



(a) $t = 4[s]$ (b) $t = 6[s]$

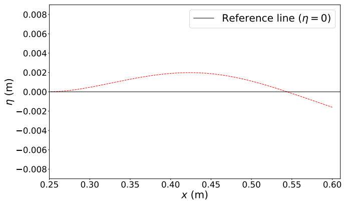


(c) $t = 8[s]$ (d) $t = 10[s]$

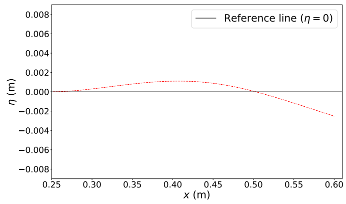


(e) $t = 12[s]$ (f) $t = 14[s]$

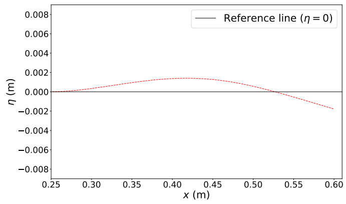
Figure 4.31 – Snapshots of the fluid velocity magnitude at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-2}$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.



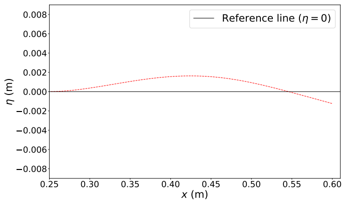
(a) $t = 4[s]$



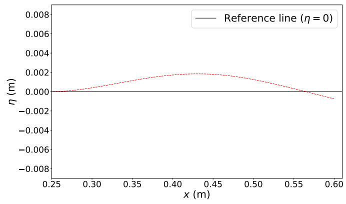
(b) $t = 6[s]$



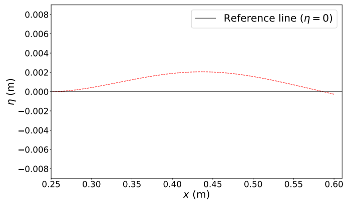
(c) $t = 8[s]$



(d) $t = 10[s]$



(e) $t = 12[s]$



(f) $t = 14[s]$

Figure 4.32 – Snapshots of the deflection of the structure (dashed red line) at different time instants corresponding to the rigidity coefficient $\alpha = 10^{-2}$ and perturbation parameter $\boldsymbol{\beta = 1.5}$.

$\underline{\boldsymbol{\alpha = 10^{-3}}}$. In this experiment, we fix the rigidity coefficient $\alpha = 10^{-3}$. Figure 4.33 shows the snapshot of the magnitude of the fluid velocity at the time instant $t = 6[s]$. In Figure 4.34, we show the meshing around the reentrant corners. In those figures we can observe the mesh deterioration around the lower reentrant corner. As pointed out at the beginning of this chapter, it is known that the classical harmonic extension used to define the ALE mapping performs well when the displacement of the structure is small (see, for instance, [Wic11]). This suggests investigating alternative approaches that allow us to carry out numerical simulations where the displacement of the structure is not small. In this regard, some techniques to be explored include those proposed in [Wic11] and [HC23], for instance.

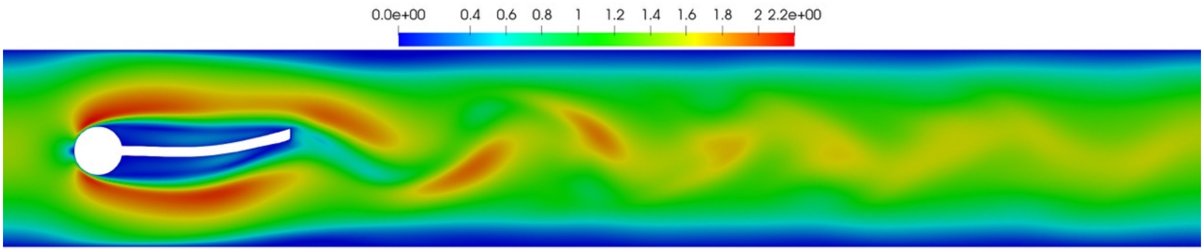
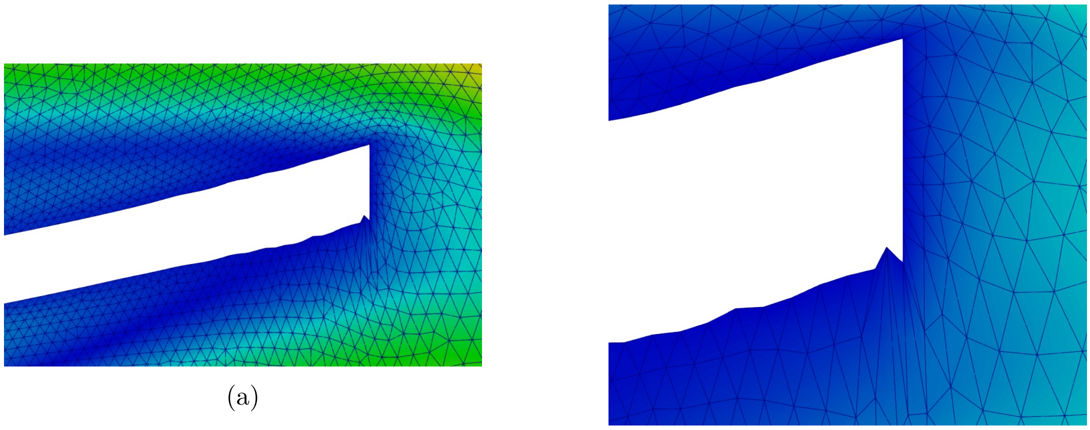


Figure 4.33 – Snapshots of the fluid velocity magnitude at the time instant $t = 6[s]$ corresponding to the rigidity coefficient $\alpha = 10^{-3}$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.



(a) (b)

Figure 4.34 – Snapshots of the mesh around the reentrant corners at $t = 6[s]$, corresponding to the rigidity coefficient $\alpha = 10^{-3}$ and perturbation parameter $\boldsymbol{\beta = 0.5}$.

Appendix A: Terms to be linearized

In this appendix, we present all the terms to be linearized around the stationary solution $(\mathbf{u}_s, p_s, 0, 0)$. First, we introduce the mapping that allows us to rewrite the system (4.2.3) in the fixed reference domain Ω. Let $X(t,\cdot) : \Omega \longrightarrow \Omega_{\eta(t)}$ defined by

$$X(t,\cdot) = I + \mathbf{d}(t,\cdot),$$

where $\mathbf{d}$ is the solution of the elliptic equation

$$\Delta \mathbf{d} = 0 \ \text{ in } \Omega, \ \ \mathbf{d} = \eta_1 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \mathbf{d} = 0 \ \text{ on } \Gamma \setminus \Gamma_s. \tag{4.4.6}$$

We denote by J the jacobian matrix of the transformation X.

• **Momentum equation**. After using [HP00, Proposition 5.4.3 p. 190] to rewrite the boundary terms, we obtain

$$\begin{aligned}\int_\Omega \partial_t \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} |J| = &- \int_\Omega \left(J^{-1}(\hat{\mathbf{u}} - \mathbf{w}) \cdot \nabla \right) \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} |J| + \int_\Omega \hat{p} \mathbb{I}_{2\times 2} : (\nabla \hat{\mathbf{v}} J^{-1}) |J| \\ &- \int_\Omega \left(\nu \left(\nabla \hat{\mathbf{u}} J^{-1} + J^{-\top} (\nabla \hat{\mathbf{u}})^\top \right) \right) : (\nabla \hat{\mathbf{v}} J^{-1}) |J| + \int_{\Gamma_{top}} \widehat{\boldsymbol{\lambda}}_{top} \cdot \hat{\mathbf{v}} \| J^{-\top} \vec{\mathbf{e}}_2 \| |J| \\ &+ \int_{\Gamma_{bot}} \widehat{\boldsymbol{\lambda}}_{bot} \cdot \hat{\mathbf{v}} \| J^{-\top} \vec{\mathbf{e}}_2 \| |J| + \int_{\Gamma_{lat}} \widehat{\boldsymbol{\lambda}}_{lat} \cdot \hat{\mathbf{v}} \| J^{-\top} \vec{\mathbf{e}}_1 \| |J| + \int_{\Gamma_{fix}} \boldsymbol{\lambda}_{fix} \cdot \hat{\mathbf{v}}, \ \text{ for all } \hat{\mathbf{v}}.\end{aligned} \tag{4.4.7}$$

• **Conservation equation**.

$$\int_\Omega \nabla \hat{\mathbf{u}} : J^{-\top} \widehat{\Psi} |J| = 0, \ \text{ for all } \widehat{\Psi}. \tag{4.4.8}$$

• **Kinematic equation**.

$$\begin{aligned}&\int_{\Gamma_{top}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}} \| J^{-\top} \vec{\mathbf{e}}_2 \| |J| - \int_0^{\ell_s} \eta_2 \vec{\mathbf{e}}_2 \cdot \hat{\boldsymbol{\tau}} \sqrt{1 + \eta_{1,x}^2} = 0, \ \text{ for all } \hat{\boldsymbol{\tau}}, \\ &\int_{\Gamma_{bot}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}} \| J^{-\top} \vec{\mathbf{e}}_2 \| |J| - \int_0^{\ell_s} \eta_2 \vec{\mathbf{e}}_2 \cdot \hat{\boldsymbol{\tau}} \sqrt{1 + \eta_{1,x}^2} = 0, \ \text{ for all } \hat{\boldsymbol{\tau}}, \\ &\int_{\Gamma_{lat}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}} \| J^{-\top} \vec{\mathbf{e}}_1 \| |J| - \int_{\Gamma_{lat}} \eta_2 \vec{\mathbf{e}}_2 \cdot \hat{\boldsymbol{\tau}} \| J^{-\top} \vec{\mathbf{e}}_1 \| |J| = 0, \ \text{ for all } \hat{\boldsymbol{\tau}}.\end{aligned} \tag{4.4.9}$$

Let us notice that in the third identity in (4.4.9) can be rewritten as

$$\int_{\Gamma_{lat}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}} - \int_{\Gamma_{lat}} \eta_2 \vec{\mathbf{e}}_2 \cdot \hat{\boldsymbol{\tau}} = 0. \tag{4.4.10}$$

Indeed, since

$$X(t, z_1, z_2) = (z_1 + d_1(t, z_1, z_2), z_2 + d_2(t, z_1, z_2)), \quad J = \begin{pmatrix} 1 + d_{1,z_1} & d_{1,z_2} \\ d_{2,z_1} & 1 + d_{2,z_2} \end{pmatrix} \tag{4.4.11}$$

and $d_1 \equiv 0$,

$$J = \begin{pmatrix} 1 & 0 \\ d_{2,z_1} & 1 + d_{2,z_2} \end{pmatrix} \quad \text{and} \quad J^{-\top} = \frac{1}{1 + d_{2,z_2}} \begin{pmatrix} 1 + d_{2,z_2} & -d_{2,z_1} \\ 0 & 1 \end{pmatrix}. \tag{4.4.12}$$

Next,

$$J^{-\top}\vec{\mathbf{e}}_1 = \frac{1}{1+d_{2,z_2}}\begin{pmatrix} 1+d_{2,z_2} & -d_{2,z_1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.4.13}$$

Then, using (4.4.13) and the fact that $d_{2,z_2} = 0$ on Γ_{lat}, we deduce that

$$-\int_{\Gamma_{lat}} \eta_2\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}\|J^{-\top}\vec{\mathbf{e}}_1\||J| = -\int_{\Gamma_{lat}} \eta_2\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}\|\vec{\mathbf{e}}_1\||1+d_{2,z_2}| = -\int_{\Gamma_{lat}} \eta_2\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}. \tag{4.4.14}$$

Moreover,this last expression can be further simplified as follows. We denote by $(\zeta_j, \widetilde{\zeta}_j)_{1\le j\le N_s}$ the Hermite basis and by $(\hat{\boldsymbol{\tau}}_j)_{1\le j\le N_{m\ell}}$ the $\mathbb{P}^1$ basis used. Then, for all $i = 1, \dots N_{m\ell}$,

$$\begin{aligned} -\int_{\Gamma_{lat}} \eta_2\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}_i &= -\int_{\Gamma_{lat}} \eta_2(\ell_s)\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}_i \\ &= -\sum_{j=1}^{N_s}\int_{\Gamma_{lat}} \left(\eta_2^j\zeta_j + \eta_{2,x}^j\widetilde{\zeta}_j\right)\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}_i \\ &= \underbrace{-\sum_{j=1}^{N_s-1}\int_{\Gamma_{lat}} \left(\eta_2^j\zeta_j + \eta_{2,x}^j\widetilde{\zeta}_j\right)\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}_i}_{=0} - \int_{\Gamma_{lat}} \left(\eta_2^{N_s}\zeta_{N_s} + \eta_{2,x}^{N_s}\widetilde{\zeta}_{N_s}\right)\vec{\mathbf{e}}_2\cdot\hat{\boldsymbol{\tau}}_i. \end{aligned} \tag{4.4.15}$$

- **Structure equations**.

$$\begin{aligned} &\int_0^{\ell_s} \eta_{1,t}\zeta = \int_0^{\ell_s} \eta_2\zeta \ \text{ for all } \zeta, \\ &\int_0^{\ell_s} \eta_{2,t}\zeta = -\alpha\int_0^{\ell_s} \Delta\eta_1\Delta\zeta - \gamma\int_0^{\ell_s} \Delta\eta_2\Delta\zeta \\ &-\int_0^{\ell_s} \hat{\boldsymbol{\lambda}}_{top}\cdot\vec{\mathbf{e}}_2\zeta\sqrt{1+\eta_{1,x}^2} - \int_0^{\ell_s} \hat{\boldsymbol{\lambda}}_{bot}\cdot\vec{\mathbf{e}}_2\zeta\sqrt{1+\eta_{1,x}^2}, \ \text{ for all } \zeta. \end{aligned} \tag{4.4.16}$$

- **Equations for w and d**.

$$\left\{\begin{aligned} &\int_\Omega \nabla\mathbf{w} : \nabla\boldsymbol{\varphi} - \int_{\Gamma_s} \boldsymbol{\lambda}_{s,w}\cdot\boldsymbol{\varphi} - \int_{\Gamma\backslash\Gamma_s} \boldsymbol{\lambda}_{f,w}\cdot\boldsymbol{\varphi} = 0, \ \text{ for all } \boldsymbol{\varphi}, \\ &\int_{\Gamma_s} \mathbf{w}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s} \eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0, \quad \text{for all } \boldsymbol{\tau}_s, \\ &\int_{\Gamma\backslash\Gamma_s} \mathbf{w}\cdot\boldsymbol{\tau}_f = 0, \ \text{ for all } \boldsymbol{\tau}_f, \end{aligned}\right. \tag{4.4.17}$$

and

$$\left\{\begin{aligned} &\int_\Omega \nabla\mathbf{d} : \nabla\boldsymbol{\varphi} - \int_{\Gamma_s} \boldsymbol{\lambda}_{s,d}\cdot\boldsymbol{\varphi} - \int_{\Gamma\backslash\Gamma_s} \boldsymbol{\lambda}_{f,d}\cdot\boldsymbol{\varphi} = 0, \ \text{ for all } \boldsymbol{\varphi}, \\ &\int_{\Gamma_s} \mathbf{d}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s} \eta_1\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0, \quad \text{for all } \boldsymbol{\tau}_s, \\ &\int_{\Gamma\backslash\Gamma_s} \mathbf{d}\cdot\boldsymbol{\tau}_f = 0, \ \text{ for all } \boldsymbol{\tau}_f. \end{aligned}\right. \tag{4.4.18}$$

Chapter 5

Stabilization of the Fluid-Structure Interaction system

Abstract of the current chapter

In this chapter, we study the stabilization of a fluid-structure interaction system about an unstable stationary solution. We consider a fluid-structure interaction model coupling the incompressible Navier-Stokes equations in a 2D rectangular-type domain, and an elastic structure governed by a damped Euler-Bernoulli beam equation. The structure, which is assumed to be clamped at one end and free at the other, is immersed in the domain occupied by the fluid. We prove that the system is exponentially stable, locally around an unstable stationary solution, for any given decay rate, by using a feedback control corresponing to a force term in the structure equation.

We emphasize that the computation of adjoint fluid-structure operator was obtained by formal computations. In particular, the well-posedness of such a system has not been established and the corresponding analysis will be the subject of a future work. However, this analysis is not needed to prove the main result of the chapter.

Outline of the current chapter

5.1 Introduction **149**

5.2 Notation and statement of the main results **152**

5.2.1 Notation 152

5.2.2 System in the reference configuration and statement of the main results 154

5.3 Analysis of the fluid-structure operator **158**

5.3.1 Fluid operator 158

5.3.2 Structure operator 164

5.3.3 Fluid-Structure operator 165

5.4 Characterization of the adjoint operator $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ **166**

5.5 Resolvent of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and analyticity of the underlying semigroup **171**

5.6 Eigenvalue problems **173**

5.6.1 Direct eigenvalue problem 173

5.6.2 Adjoint eigenvalue problem 174

5.7 Stabilization of the linearized system **175**

5.7.1 Stabilizability of $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ 178

5.7.2 Feedback control law 180

5.8 Stabilization of the nonhomogeneous linearized system **181**
5.9 Stabilization of the nonlinear system **188**
5.9.1 Estimate of nonlinear terms 188
5.9.2 Proof of Theorem 5.2.1 192
Appendix A: Nonlinear terms coming from the geometric transformation (5.2.8) **195**
Appendix B: Proof of Theorem 5.3.2 **197**
Appendix C: Justification of the identity (5.4.14) **(formal computations)** **199**
Appendix D: Proof of Proposition 5.7.1 **204**
Appendix E: Proof of Proposition 5.7.3 **206**

5.1 Introduction

In this chapter, we are interested in determining a feedback control law of finite dimension able to stabilize a fluid-structure interaction system in a neighborhood of an unstable stationary solution. Before introducing the governing equations, let us introduce some notation.

The reference configuration Ω of the fluid domain is given by

$$\Omega = [-L/2, L] \times [-\ell, \ell] \setminus \mathcal{S},$$

where $\mathcal{S} = \mathcal{S}_r \cup \mathcal{S}_e$. The boundary Γ of Ω is divided into

$$\Gamma = \Gamma_i \cup \Gamma_r \cup \Gamma_s \cup \Gamma_w \cup \Gamma_n,$$

where

$$\begin{aligned}
\Gamma_i &= \{-L/2\} \times [-\ell, \ell], \\
\Gamma_r &= \{(r(\cos(\theta) - \cos(\theta_0)), r\sin(\theta)) \mid \theta \in [\theta_0, 2\pi - \theta_0]\}, \quad r > 0, \theta_0 \in (0, \pi/2), \\
\Gamma_s &= \Gamma_s^- \cup \Gamma_s^+ \cup \Gamma_s^\ell, \\
\Gamma_w &= [-L/2, L] \times \{-\ell\} \cup [-L/2, L] \times \{\ell\}, \\
\Gamma_n &= \{L\} \times [-\ell, \ell],
\end{aligned}$$

with $\Gamma_s^- = [0, \ell_s] \times \{-e\}$, $\Gamma_s^+ = [0, \ell_s] \times \{e\}$ and $\Gamma_s^\ell = \{\ell_s\} \times [-e, e]$. We also set $\Gamma_d = \Gamma \setminus \Gamma_n$. See Figure 5.1.

For a given function η defined from $(0, \infty) \times (0, \ell_s)$ to $\mathbb{R}$ that describes the centerline displacement of the elastic part of the structure, we denote by $\Omega_{\eta(t)}$ the fluid domain at time t and by $\Gamma_{\eta(t)} = \Gamma_{\eta(t)}^- \cup \Gamma_{\eta(t)}^+ \cup \Gamma_{\eta(t)}^\ell$ the fluid-structure interface, where $\Gamma_{\eta(t)}^-$ and $\Gamma_{\eta(t)}^+$ represent the bottom and top of the elastic part of the structure, respectively and $\Gamma_{\eta(t)}^\ell$ the lateral part (see Figure 5.2). Here, $\Gamma_{\eta(t)}^-$, $\Gamma_{\eta(t)}^+$ and $\Gamma_{\eta(t)}^\ell$ are given by

$$\Gamma_{\eta(t)}^- = \{(x, \eta(t,x) - e) | \; x \in [0, \ell_s]\}, \; \Gamma_{\eta(t)}^+ = \{(x, \eta(t,x) + e) | \; x \in [0, \ell_s]\}$$

and

$$\Gamma_{\eta(t)}^\ell = \{(\ell_s, y) \mid y = (1-\lambda)(-e + \eta(t, \ell_s)) + \lambda(e + \eta(t, \ell_s))), \quad \lambda \in [0,1]\}.$$

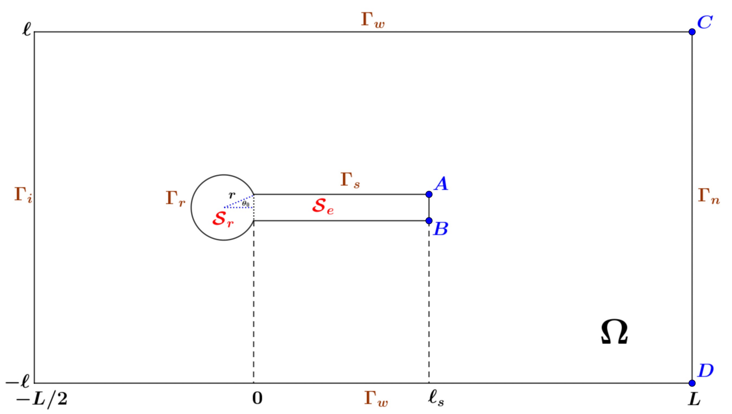


Figure 5.1 – Reference configuration.

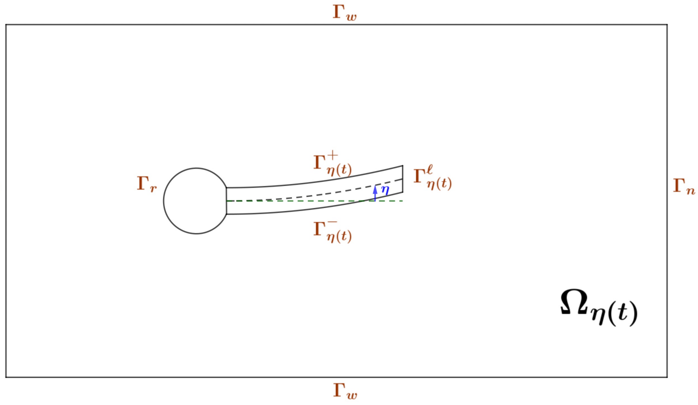


Figure 5.2 – Physical domain. The dashed lines denotes the reference centerline.

We set

$$
\begin{aligned}
&Q^\infty_\eta = \bigcup_{t\in(0,\infty)} \left(\{t\}\times\Omega_{\eta(t)}\right),\ \Sigma^\infty_\eta = \bigcup_{t\in(0,\infty)} \left(\{t\}\times\Gamma_{\eta(t)}\right),\\
&Q^\infty = (0,\infty)\times\Omega,\ \ \Sigma^\infty_s = (0,\infty)\times\Gamma_s,\\
&\Sigma^\infty_i = (0,\infty)\times\Gamma_i,\ \ \Sigma^\infty_d = (0,\infty)\times\Gamma_d,\\
&\Sigma^\infty_r = (0,\infty)\times\Gamma_r,\ \ \Sigma^\infty_n = (0,\infty)\times\Gamma_n.
\end{aligned}
$$

The governing equations for the fluid-structure interaction system are given by

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q^\infty_\eta, \tag{5.1.1a}$$
$$\operatorname{div}\mathbf{u} = 0 \ \text{ in } Q^\infty_\eta, \tag{5.1.1b}$$
$$\mathbf{u} = \mathbf{g}_s + \mathbf{g}_p \ \text{ on } \Sigma^\infty_i,\ \ \mathbf{u} = 0 \ \text{ on } \Sigma^\infty_r\cup\Sigma^\infty_w, \tag{5.1.1c}$$
$$\mathbf{u} = \eta_t\vec{\mathbf{e}}_2 \ \text{ on } \Sigma^\infty_\eta,\ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma^\infty_n, \tag{5.1.1d}$$
$$\mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega, \tag{5.1.1e}$$
$$\partial^2_t\eta + \alpha\Delta^2_s\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) + f_s + f \text{ in } (0,\infty)\times(0,\ell_s), \tag{5.1.1f}$$
$$\eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ on } (0,\infty)\times\{0\}, \tag{5.1.1g}$$
$$\partial^2_{x_1}\eta = 0 \ \text{ and } \ \partial^3_{x_1}\eta = 0 \ \text{ on } (0,\infty)\times\{\ell_s\}, \tag{5.1.1h}$$
$$\eta(0) = 0 \ \text{ and } \ \partial_t\eta(0) = \eta^0_2 \ \text{ in } (0,\ell_s), \tag{5.1.1i}$$

where $\mathbf{u}$ and p respresent the fluid velocity and pressure. Here, $\sigma(\mathbf{u},p)$ is the Cauchy stress tensor given by

$$\sigma(\mathbf{u},p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

with $\nu > 0$ denoting the fluid viscosity. The inflow boundary condition $\mathbf{g}_s$ is time-independent, while $\mathbf{g}_p$ is a time-dependent perturbation of $\mathbf{g}_s$. The elastic part of the structure is governed by the reference centerline curve η. The parameters $\alpha > 0$ and $\gamma > 0$ are constants relative to the structure. The damping operator B is given by

$$B = (\Delta^2_s)^{1/2},\ \ D(B) = \left\{\eta\in H^2(0,\ell_s) \mid \eta(0) = \partial_{x_1}\eta(0) = 0\right\},$$

where $\Delta^2_s = \partial^4_{x_1}$, with $\mathcal{D}(\Delta^2_s) = H^4_{\{0,\ell_s\}}(0,\ell_s)$. The expression of the force exerted by the fluid on $\Gamma^+_{\eta(t)}\cup\Gamma^-_{\eta(t)}$ is given by

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}^+_{\eta(t)} + \sigma^-(\mathbf{u},p)\mathbf{n}^-_{\eta(t)}\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2, \tag{5.1.2}$$

where

$$\sigma^{\pm}(\mathbf{u},p)=\sigma(\mathbf{u}(t,x,\eta(t,x)\pm e),p(t,x,\eta(t,x)\pm e)),$$

and $\mathbf{n}^{+}_{\eta(t)}$ (resp. $\mathbf{n}^{+}_{\eta(t)}$) is the unit normal vector to $\Gamma^{+}_{\eta(t)}$ (resp. $\Gamma^{-}_{\eta(t)}$) exterior to $\Omega_{\eta(t)}$.

We assume that the control function f has the form

$$f(t,x,y)=\sum_{i=1}^{N_c} f_i(t)w_i(x,y), \tag{5.1.3}$$

where the functions $(w_i)_{1\le i\le N_c}$ are chosen appropriately. We will discuss their choice later.

Let $(\mathbf{u}_s,p_s)$ be a solution of the stationary Navier-Stokes equations

$$\begin{cases}(\mathbf{u}_s\cdot\nabla)\mathbf{u}_s-\operatorname{div}\sigma(\mathbf{u}_s,p_s)=0, \ \ \operatorname{div}\mathbf{u}_s=0 \ \text{ in } \Omega,\\ \mathbf{u}_s=\mathbf{g}_s, \ \text{ on } \Gamma_i, \ \ \mathbf{u}_s=0 \ \text{ on } \Gamma\setminus\Gamma_i,\\ \sigma(\mathbf{u}_s,p_s)\mathbf{n}=0 \ \text{ on } \Gamma_n.\end{cases} \tag{5.1.4}$$

The regularity required of $\mathbf{u}_s$ and p_s is specified in A1. Let us consider the time-independent function f_s chosen in such a way that the triplet $(\mathbf{u},\eta,\eta_t)=(\mathbf{u}_s,0,0)$ constitutes a stationary solution of system (5.1.1). We assume that it is an unstable stationary solution. The aim of this chapter is to find a control $\mathbf{f}=(f_j)_{j=1}^{N_c}$ given in a feedback form, able to stabilize the system (5.1.1) with a prescribed exponential decay rate $\omega\ge 0$, locally about $(\mathbf{u},\eta,\eta_t)=(\mathbf{u}_s,0,0)$.

We briefly describe below the approach adopted, along with the main difficulties encountered. The strategy consists first in stabilizing the linerized system. Then, using the feedback law obtained in this step, we show that it also stabilizes the nonlinear system, provided that appropriate assumptions on the initial and boundary data are satisfied.

Before linealizing the system, and similarly to the analysis carried out in Chapter 3 concerning the existence of strong solutions, it is necessary to rewrite the system (5.1.1) on a fixed reference domain by introducing an appropriate change of variables. As a consequence, the difficulties discussed in Chapter 3 persist at this stage of the analysis. We briefly discuss these aspects below.

• *Analysis of the stationary Oseen system*. We have to study the regularity of $\mathbf{w}$ and pressure π of the stationary Oseen system

$$\begin{cases}-\operatorname{div}\sigma(\mathbf{w},\pi)+(\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s=\mathbf{F} \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{w}=h \ \text{ in } \Omega,\\ \mathbf{w}=\mathbf{g} \ \text{ on } \Gamma_i, \ \ \sigma(\mathbf{w},\pi)\mathbf{n}=0 \ \text{ on } \Gamma_n,\end{cases} \tag{5.1.5}$$

where $\mathbf{F}$, h and $\mathbf{g}$ are stationary data. In this regard, under appropriate conditions on $(\mathbf{u}_s,p_s)$, we adapted the regularity result presented in Theorem 3.3.1 of Chapter 3.

• *Analysis of the instationary Oseen system*. An important step in the analysis consists in studying the system

$$\begin{cases}\dfrac{\partial\mathbf{w}}{\partial t}-\operatorname{div}\sigma(\mathbf{w},\pi)+(\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s=\mathbf{F} \ \text{ in } (0,T)\times\Omega,\\ \operatorname{div}\mathbf{w}=h \ \text{ in } (0,T)\times\Omega,\\ \mathbf{w}=\mathbf{g} \ \text{ on } (0,T)\times\Gamma_d, \ \ \sigma(\mathbf{w},\pi)\mathbf{n}=0 \ \text{ on } (0,T)\times\Gamma_n,\\ \mathbf{w}(0)=\mathbf{w}_0 \ \text{ in } \Omega,\end{cases} \tag{5.1.6}$$

and establishing the analyticity of the semigroup associated with the Oseen operator in the context of heterogeneous Sobolev spaces. The ideas presented in Subsection 3.3.4 of Chapter 3, can be suitably adapted to address the case of the Oseen operator.

On the other hand, additional issues arise, which are intrisic to the stabilization of the linearized system. Although the difficulties outlined below have been already treated in [FNR19], it is important to highlight them, since they are also encountered in our analysis.

• *Analysis of eigenvalue problems*. Since the system (5.1.1) is linearized around a nontrivial stationary solution, the analysis of both the direct and the adjoint eigenvalues problems involves non-standard conditions. Specifically, the direct eigenvalue problem requires handling an algebraic constraint of the form

$$\operatorname{div} \mathbf{v} = A_3 \eta_1 \text{ in } \Omega, \quad \mathbf{v} = \eta_2 \vec{\mathbf{e}}_2 \text{ on } \Gamma_s, \tag{5.1.7}$$

while the adjoint eigenvalue problem features a condition of the type

$$\operatorname{div} \mathbf{\Phi} = 0 \text{ in } \Omega, \quad \mathbf{v} = \zeta_2 \vec{\mathbf{e}}_2 \text{ on } \Gamma_s, \tag{5.1.8}$$

where A_3 is linear bounded operator.

• *Equivalence between PDE and operator formulations*. Closely related to the difficulty mentioned above is the necessity, in the spectral study of the linearized system, to establish the equivalence between the partial differential equation formulations and the operator formulation of both the direct and adjoint eigevanlue problems.

5.2 Notation and statement of the main results

5.2.1 Notation

Usual and weighted Sobolev spaces

We recall some of the notation introduced in Chapter 3. We set $\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2)$ and $\mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2)$ for $s > 0$. We also introduce the following functional spaces:

$$\begin{aligned}
&\mathbf{H}^s_{\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{H}^s(\Omega) \mid \mathbf{u} = 0 \text{ on } \Gamma_d\} \text{ for } s > 1/2,\\
&\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \ \ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_d\},\\
&\mathbf{V}^1_{\Gamma_d}(\Omega) = \mathbf{H}^1_{\Gamma_d}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega),\\
&H^1_{\{0\}}(0,\ell_s) = \{\mu \in H^1(0,\ell_s) \mid \mu(0) = 0\},\\
&H^2_{\{0\}}(0,\ell_s) = \{\mu \in H^2(0,\ell_s) \mid \mu(0) = \partial_{x_1}\mu(0) = 0\},\\
&H^3_{\{0,\ell_s\}}(0,\ell_s) = \{\mu \in H^3(0,\ell_s) \cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s) = 0\},\\
&H^4_{\{0,\ell_s\}}(0,\ell_s) = \{\mu \in H^4(0,\ell_s) \cap H^2_{\{0\}}(0,\ell_s) \mid \ \partial^2_{x_1}\mu(\ell_s) = \partial^3_{x_1}\mu(\ell_s) = 0\},\\
&H^{2,1}_{\{0\}}((0,T)\times(0,\ell_s)) = L^2(0,T;H^2_{\{0\}}(0,\ell_s)) \cap H^1(0,T;L^2(0,\ell_s)),\\
&H^{4,2}_{\{0,\ell_s\}}((0,T)\times(0,\ell_s)) = L^2(0,T;H^4_{\{0,\ell_s\}}(0,\ell_s)) \cap H^2(0,T;L^2(0,\ell_s)).
\end{aligned}$$

All of the previous spaces are endowed with the natural norms.

We introduce the space for the inflow conditions

$$\mathbf{H}(\Gamma_i) = \left\{\mathbf{g} = (g_1,g_2) \mid g_2 = 0 \ \text{ and } \ g_1 \in H^{\frac{3}{2}}(\Gamma_i) \cap H^1_0(\Gamma_i)\right\}, \tag{5.2.1}$$

equipped with the norm $(g_1, g_2) \mapsto \|g_1\|_{H^{\frac{3}{2}}(\Gamma_i)}$. For $0 < s < 1/2$, we introduce the intermediate spaces

$$\mathbf{H}^s_{\Gamma_d}(\Omega) = [\mathbf{L}^2(\Omega), \mathbf{H}^1_{\Gamma_d}(\Omega)]_s.$$

The dual of $\mathbf{H}^s_{\Gamma_d}(\Omega)$ is denoted by $\mathbf{H}^{-s}_{\Gamma_d}(\Omega)$.

Let us denote by $\mathcal{J}$ the set of vertices of Γ. For $\beta > 0$, we introduce the norms

$$\begin{aligned}
\|\mathbf{w}\|_{\mathbf{H}^2_\beta} &:= \left(\sum_{|k|=0}^{2} \sum_{i=1}^{2} \int_\Omega \Big(\prod_{J \in \mathcal{J}} r_J^{2\beta} \Big) |\partial^k w_i|^2 dx \right)^{1/2}, \quad \mathbf{w} \in C^\infty(\overline{\Omega}; \mathbb{R}^2), \\
\|p\|_{H^1_\beta} &:= \left(\sum_{|k|=0}^{1} \int_\Omega \Big(\prod_{J \in \mathcal{J}} r_J^{2\beta} \Big) |\partial^k p|^2 dx \right)^{1/2}, \quad p \in C^\infty(\overline{\Omega}; \mathbb{R}),
\end{aligned} \tag{5.2.2}$$

where r_J stands for the distance to a junction point $J \in \mathcal{J}$, $k = (k_1, k_2) \in \mathbb{N}^2$ denotes a two-index with length $|k| = k_1 + k_2$, ∂_k denotes the corresponding partial differential operator and $\mathbf{w} = (w_1, w_2)$. We denote by $\mathbf{H}^2_\beta(\Omega; \mathbb{R}^2)$ (respectively, $H^1_\beta(\Omega)$) the closure of $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ (respectively, $C^\infty(\overline{\Omega})$) in the norm $\|\cdot\|_{\mathbf{H}^2_\beta(\Omega)}$ (respectively, $\|\cdot\|_{H^1_\beta(\Omega)}$).

Throughout this chapter, we will use the following regularity exponents

$$\alpha^* \in (0, 1/2) \quad \text{and} \quad \delta^* \in (0, 1/2). \tag{5.2.3}$$

Heterogeneous Sobolev spaces

Let $\varepsilon > 0$. We introduce the cut-off function $\Psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $0 \leq \Psi \leq 1$,

$$\Psi = 1 \text{ on } (-L/2, \ell_s + \varepsilon/2) \times (-\ell, \ell) \text{ and } \Psi = 0 \text{ on } (\ell_s + \varepsilon, L) \times (-\ell, \ell). \tag{5.2.4}$$

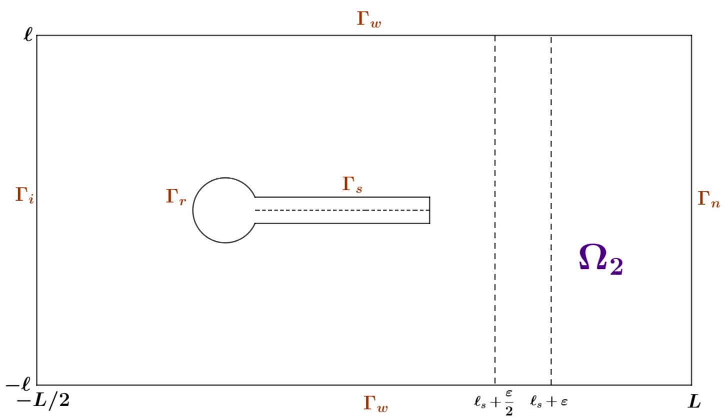


Figure 5.3 – Decomposition of the domain Ω.

We also define $\Omega_2 = (\ell_s + \varepsilon/2, L) \times (-\ell, \ell)$. We now recall the heterogeneous Sobolev spaces introduced in Chapter 3:

$$\begin{aligned}
\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) &= \left\{ \mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{F} \in \mathbf{L}^2(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1(\Omega) \right\}, \\
H^{\frac{1}{2}+\alpha,1}_\delta(\Omega) &= \left\{ p \in H^{\frac{1}{2}+\alpha}(\Omega) | (1-\Psi)p \in H^1_\delta(\Omega) \right\}, \\
\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) &= \left\{ \mathbf{v} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) | (1-\Psi)\mathbf{v} \in \mathbf{H}^2_\delta(\Omega) \right\},
\end{aligned} \tag{5.2.5}$$

which are respectively endowed with the natural norms.

5.2.2 System in the reference configuration and statement of the main results

This subsection is devoted to rewrite the system (5.1.1) in the reference configuration. We begin by recalling some results stated in Chapter 3.

The spatial variable in the physical domain is denoted by $x = (x_1, x_2)$, while the spatial variable in the reference configuration will be denoted by $z = (z_1, z_2)$. Let us now introduce an appropriate extension of any function defined on $[0, \ell_s]$ to $[-L/2, L]$. We set $\eta \mapsto \mathcal{E}\eta$, where

$$\mathcal{E}\eta(x_1) = \begin{cases} 0 & \text{if } x_1 \in [-L/2, 0], \\ \eta(x_1) & \text{if } x_1 \in [0, \ell_s], \\ (3\eta(2\ell_s - x_1) - 2\eta(3\ell_s - 2x_1))\,\theta(x_1) & \text{if } x_1 \in [\ell_s, L], \end{cases} \tag{5.2.6}$$

where $\theta \in C^\infty([\ell_s, L])$ is a nonnegative function with values in $[0, 1]$, which is equal to 1 in $[\ell_s, \ell_s + \varepsilon/4]$ and to 0 in $[\ell_s + \varepsilon/2, L]$, for some $0 < \varepsilon < (L - \ell_s)/2$. The following Proposition is a consequence of the definition (5.2.6).

Proposition 5.2.1. *For all $\eta \in H^{4,2}_{\{0,\ell_s\}}((0, \infty) \times (0, \ell_s))$, the following assertions are satisfied.*

(*i*) *For all $0 < a_0 < 1/2$,*

$$\mathcal{E}\eta \in L^2(0, \infty; H^{2+a_0}(-L/2, L)) \cap H^2(0, \infty; L^2(-L/2, L)).$$

and

$$\mathcal{E}\eta \in H^{\frac{1+2a_0-2\tau}{2+a_a}}(0, \infty; H^{\frac{3}{2}+\tau}(-L/2, L)), \quad \textit{for all } \tau \in (0, 3a_0/4).$$

In particular, $(\mathcal{E}\eta)(t, \cdot)$ is a map of class $\mathcal{C}^1$.

(*ii*) *For all $0 < \varepsilon_0 < L$,*

$$(\mathcal{E}\eta)|_{(\varepsilon_0, L)} \in L^2(0, \infty; H^4(\varepsilon_0, L)).$$

(*iii*) $(\mathcal{E}\eta)|_{[\ell_s+\varepsilon/2, L]} \equiv 0.$

For a given $\eta \in H^{4,2}_{\{0,\ell_s\}}((0, \infty) \times (0, \ell_s))$, we set

$$\eta^+(t, \cdot, e) := (\mathcal{E}\eta)(t, \cdot) \text{ on } [-L/2, L] \quad \text{and} \quad \eta^-(t, \cdot, -e) := (\mathcal{E}\eta)(t, \cdot) \text{ on } [-L/2, L].$$

We now introduce two C^∞ functions χ^+ and χ^-, with values in $[0, 1]$, such that $\chi^+(z_2) = 1$ in $[\frac{e}{2}, \ell]$ and $\chi^+(z_2) = 0$ in $[-\ell, \frac{e}{3}]$, $\chi^-(z_2) = 1$ in $[-\ell, -\frac{e}{2}]$ and $\chi^-(z_2) = 0$ in $[-\frac{e}{3}, \ell]$.

We introduce the set

$$\begin{aligned} E(0, \infty) = \Big\{\eta \in & H^{4,2}_{\{0,\ell_s\}}((0, \infty) \times (0, \ell_s)) \text{ such that} \\ & \min\{\ell - e + \eta^\pm_\chi(t, z) \mid (t, z) \in [0, \infty) \times \overline{\Omega}\} \geq (\ell - e)/2\Big\}. \end{aligned} \tag{5.2.7}$$

where

$$\eta^\pm_\chi(t, z) := (\mp\chi^\pm + (\ell \mp z_2)\partial_{z_2}\chi^\pm)\,\eta^\pm(t, z_1), \text{ with } (t, z) \in [0, \infty) \times \overline{\Omega}.$$

For η belonging to $E(0, \infty)$, we consider the map $X(t, \cdot) : \Omega \longrightarrow \Omega_{\eta(t)}$ defined by $X(t, z) = x = (x_1, x_2)$, where

$$\begin{cases} x_1 = z_1, \\ x_2 = \chi^\pm(z_2)\dfrac{z_2(\ell - e \mp \eta^\pm(t, z_1)) + \ell\eta^\pm(t, z_1)}{\ell - e} + (1 - \chi^\pm(z_2))z_2, \end{cases} \tag{5.2.8}$$

for all $z = (z_1, z_2) \in \Omega$. The following proposition is a consequence of the definition of the mapping X given in (5.2.8) and Proposition 5.2.1.

Proposition 5.2.2. *For all $\eta \in E(0,\infty)$, the mapping X defined in (5.2.8) satisfies the following properties:*

(i) $X(0,\Omega) = \Omega$.

(ii) *For all $t \in [0,\infty)$, we have that $X(t,\Gamma_i) = \Gamma_i$, $X(t,\Gamma_w) = \Gamma_w$, $X(t,\Gamma_r) = \Gamma_r$, $X(t,\Gamma_s) = \Gamma_{\eta(t)}$ and $X(t,\Gamma_n) = \Gamma_n$.*

(iii) *$X \in L^2(0,\infty;\mathbf{H}^{2+a_0}(\mathcal{O}_L))$ for all $0 < a_0 < 1/2$, and $X \in L^2(0,\infty;\mathbf{H}^4(\mathcal{O}_R))$.*

(iv) *$X \in H^2(0,\infty;\mathbf{L}^2(\Omega))$.*

(v) *$X(t,\cdot)$ is a $\mathcal{C}^1-$diffeomorphism from Ω onto $\Omega_{\eta(t)}$.*

We will denote by Y the inverse of X. We also set $J(t,z) = (J^{i,j})_{1\le i,j\le 2} = (\nabla X)^{-1}(t,z)$ for all $(t,z) \in (0,T)\times\Omega$. Let us notice that for X defined in (5.2.8)

$$\det(J) = \frac{\ell - e}{\ell - e + \eta_X^{\pm}(t,z)}. \tag{5.2.9}$$

In order to transform the system (5.1.1) in the reference configuration, we introduce the change of unknowns

$$\begin{aligned}
&\widehat{\mathbf{u}}(t,\mathbf{z}) = e^{\omega t}\left(\mathbf{u}(t,X(t,\mathbf{z})) - \mathbf{u}_s(\mathbf{z})\right), \ \widehat{p}(t,\mathbf{z}) = e^{\omega t}\left(p(t,X(t,\mathbf{z})) - p_s(\mathbf{z})\right),\\
&\widehat{\eta}_1(t,z_1) = e^{\omega t}\eta(t,z_1), \ \ \widehat{\eta}_2(t,z_1) = e^{\omega t}\partial_t\eta(t,z_1), \ \ \widehat{\mathbf{f}}(t) = (\widehat{f}_i(t))_{1\le i\le N_c} := e^{\omega t}\mathbf{f}(t),\\
&\widehat{\eta}_1^{\pm}(t,z_1) = e^{\omega t}\eta^{\pm}(t,z_1), \ \ \widehat{\eta}_2^{\pm}(t,z_1) = e^{\omega t}\partial_t\eta^{\pm}(t,z_1),\\
&\widehat{\mathbf{g}}_p(t,z) = e^{\omega t}\mathbf{g}_p(t,z), \ \ \widehat{\mathbf{u}}^0 = \mathbf{u}^0 - \mathbf{u}_s,
\end{aligned} \tag{5.2.10}$$

for all $(t,\mathbf{z}) \in (0,\infty)\times\Omega$. Here, $\omega > 0$. Thus, after doing the change of variable (5.2.10) we get that $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2)$ satisfies the system

$$\begin{cases}
\partial_t\widehat{\mathbf{u}} - \operatorname{div}\sigma(\widehat{\mathbf{u}},\widehat{p}) + (\mathbf{u}_s\cdot\nabla)\widehat{\mathbf{u}} + (\widehat{\mathbf{u}}\cdot\nabla)\mathbf{u}_s - A_1\widehat{\eta}_1 - A_2\widehat{\eta}_2 - \omega\widehat{\mathbf{u}} = \widehat{\mathbf{F}}_f(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}_1,\widehat{\eta}_2) \ \text{ in } Q^\infty,\\
\operatorname{div}\widehat{\mathbf{u}} = \operatorname{div}\widehat{\mathbf{G}}_{\text{div}}(\widehat{\mathbf{u}},\widehat{\eta}_1) + A_3\widehat{\eta}_1 \ \text{ in } Q^\infty,\\
\widehat{\mathbf{u}} = \widehat{\mathbf{g}}_p \ \text{ on } \Sigma_i^\infty, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_r^\infty\cup\Sigma_w^\infty,\\
\widehat{\mathbf{u}} = \widehat{\eta}_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \ \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty,\\
\widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 \ \text{ in } \Omega,\\
\partial_t\widehat{\eta}_1 - \widehat{\eta}_2 - \omega\widehat{\eta}_1 = 0 \ \text{ in } (0,\infty)\times(0,\ell_s),\\
\partial_t\widehat{\eta}_2 + \alpha\Delta_s^2\widehat{\eta}_1 + \gamma(\Delta_s^2)^{\frac12}\widehat{\eta}_2 - A_4\widehat{\eta}_1 - \omega\widehat{\eta}_2 = \gamma_s^+\widehat{p} - \gamma_s^-\widehat{p} + \widehat{F}_s(\widehat{\mathbf{u}},\widehat{\eta}_1)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \widehat{f} \ \text{ in } (0,\infty)\times(0,\ell_s),\\
\widehat{\eta}_1 = 0 \ \text{ and } \ \partial_{z_1}\widehat{\eta}_1 = 0 \ \text{ on } (0,\infty)\times\{0\},\\
\partial_{z_1}^2\widehat{\eta}_1 = 0 \ \text{ and } \ \partial_{z_1}^3\widehat{\eta}_1 = 0 \ \text{ on } (0,\infty)\times\{\ell_s\},\\
\widehat{\eta}_1(0) = 0 \ \text{ and } \ \widehat{\eta}_2(0) = \eta_2^0 \ \text{ in } (0,\ell_s).
\end{cases} \tag{5.2.11}$$

The linear operators $A_1 = (A_{1,1}, A_{1,2})$, A_2, A_3 and A_4 are defined by

$$
\begin{aligned}
A_{1,i}\widehat{\eta}_1 &= \frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}u_{s,2}u_{s,i,z_2} + \frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_{1,z_1}}{\ell-e}u_{s,1}u_{s,1,z_2} - 2\nu\frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_{1,z_1}}{\ell-e}u_{s,i,z_1z_2} \\
&\quad - 2\nu\frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}u_{s,i,z_2z_2} - \frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_{1,z_1z_1}}{\ell-e}u_{s,i,z_2} - \nu\frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}u_{s,i,z_2} \\
&\quad + \nu\frac{\widehat{\eta}^{\pm}_{\chi,z_i}}{\ell-e}u_{s,1,z_1} + \nu\frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}u_{s,1,z_1z_i} - \nu\frac{\partial}{\partial_{z_i}}\left[\frac{\ell\mp z_2}{\ell-e}\chi^{\pm}\right]\widehat{\eta}^{\pm}_{1,z_i}u_{s,1,z_2} - \nu\left[\frac{\ell\mp z_2}{\ell-e}\chi^{\pm}\right]\widehat{\eta}^{\pm}_{1,z_1z_i}u_{s,1,z_2} \\
&\quad - \nu\left[\frac{\ell\mp z_2}{\ell-e}\chi^{\pm}\right]\widehat{\eta}^{\pm}_{1,z_i}u_{s,1,z_1z_i} + \frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_{1,z_1}}{\ell-e}p_{s,z_2}\delta_{1,i} + \frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}p_{s,z_2}\delta_{2,i}, \quad i=1,2, \\
A_2\widehat{\eta}_2 &= \frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_2}{\ell-e}\mathbf{u}_{s,z_2}, \quad A_3\widehat{\eta}_1 = -\frac{\widehat{\eta}^{\pm}_{\chi}}{\ell-e}u_{s,1,z_1} + \frac{(\ell\mp z_2)\chi^{\pm}\widehat{\eta}^{\pm}_{1,z_1}}{\ell-e}u_{s,1,z_1}, \\
A_4\widehat{\eta}_1 &= -\nu\widehat{\eta}^{\pm}_{1,z_1}\gamma_s^{+,-}u_{s,1,z_2},
\end{aligned}
\tag{5.2.12}
$$

where $\delta_{i,j}$ denotes the Kronecker delta. The expressions of $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\mathrm{div}}$ and $\widehat{F}_s$ can be found in Appendix A.

The corresponding linearized system associated to (5.2.11) is given by

$$
\begin{cases}
\partial_t\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},q) + (\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega\mathbf{v} = \mathbf{F}_f \ \text{ in } Q^{\infty}, \\
\operatorname{div}\mathbf{v} = A_3\eta_1 + \operatorname{div}\mathbf{G}_{\mathrm{div}} \ \text{ in } Q^{\infty}, \\
\mathbf{v} = \mathbf{g}_p \ \text{ on } \Sigma_i^{\infty}, \ \ \mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^{\infty}, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma_r^{\infty}\cup\Sigma_w^{\infty}, \ \ \sigma(\mathbf{v},q)\mathbf{n} = 0 \ \text{ on } \Sigma_n^{\infty}, \\
\partial_t\eta_1 - \eta_2 - \omega\eta_1 = 0 \ \text{ in } (0,\infty)\times(0,\ell_s), \\
\partial_t\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac12}\eta_2 - A_4\eta_1 - \omega\eta_2 = -\gamma_s^+q + \gamma_s^-q + F_s + f \ \text{ in } (0,\infty)\times(0,\ell_s), \\
\eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ on } (0,\infty)\times\{0\} \ \text{ and } \ \partial^2_{x_1}\eta_1 = 0, \partial^3_{x_1}\eta_1 = 0 \ \text{ on } (0,\infty)\times\{\ell_s\}, \\
\eta_1(0) = 0 \ \text{ and } \ \eta_2(0) = \eta_2^0 \ \text{ in } (0,\ell_s).
\end{cases}
\tag{5.2.13}
$$

For simplicity of notation, we omit the symbol $\widehat{}$ in the preceding system. Before stating the main results of this chapter, we will introduce notation and the assumptions used throughout. Let $\alpha\in(0,\alpha^*)$ and $\delta\in(\delta^*,1)$. Let us first consider the class

$$
\begin{aligned}
&\mathbf{u}\in L^2(0,\infty;\mathbf{H}^{\frac32+\alpha,2}_{\delta}(\Omega_{\eta(\cdot)}))\cap H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega_{\eta(\cdot)})), \\
&p\in L^2(0,\infty;H^{\frac12+\alpha,1}_{\delta}(\Omega_{\eta(\cdot)})), \\
&\eta\in L^2(0,\infty;H^4_{\{0,\ell_s\}}(0,\ell_s))\cap H^2(0,\infty;L^2(0,\ell_s)).
\end{aligned}
\tag{5.2.14}
$$

We now introduce the space

$$
\begin{aligned}
\mathcal{Z}_{\infty} = \Big(L^2(0,\infty;\mathbf{H}^{\frac32+\alpha,2}_{\delta}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega))\Big)& \\
\times L^2(0,\infty;H^{\frac12+\alpha,1}_{\delta}(\Omega))\times H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s))&,
\end{aligned}
\tag{5.2.15}
$$

equipped with the norm

$$
\begin{aligned}
\|(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\|_{\mathcal{Z}_{\infty}} &= \|\widehat{\mathbf{u}}\|_{L^2(0,\infty;\mathbf{H}^{\frac32+\alpha,2}_{\delta})\cap H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0})} \\
&\quad + \|\widehat{p}\|_{L^2(0,\infty;H^{\frac12+\alpha,1}_{\delta})} + \|\widehat{\eta}\|_{H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s))},
\end{aligned}
\tag{5.2.16}
$$

where

$$
\|\widehat{\eta}\|_{H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s))} = \|\widehat{\eta}\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\widehat{\eta}_t\|_{H^{2,1}((0,\infty)\times(0,\ell_s))}.
$$

For a given $R > 0$, we also introduce the set

$$B_\infty(R, \mathbf{u}_0, \eta_2^0) := \Big\{ (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}) \in \mathcal{Z}_\infty \mid \|(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta})\|_{\mathcal{Z}_\infty} \le R, \ \ \widehat{\eta} \in E(0,\infty) \\ \text{and } \widehat{\mathbf{u}}(0) = \mathbf{u}_0,\ \widehat{\eta}(0) = 0,\ \widehat{\eta}_t(0) = \eta_2^0 \Big\}. \tag{5.2.17}$$

Assumptions.

Assumption 1. Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. We assume that $\mathbf{g}_s \in \mathbf{H}(\Gamma_i)$ and that the system (5.1.4) admits a solution $(\mathbf{u}_s, p_s) \in \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega) \times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$.

(A1)

The assumptions A2 and A3 are stated in Section 5.7. In Assumption A2, we state a condition concerning the spectrum of the adjoint of the Oseen operator and the adjoint of the structure operator. More precisely, we assume that the parts of the spectrum, contained in the half plane $\{\lambda \in \mathbb{C} \mid \Re\lambda \ge \omega\}$, of the adjoint of the Oseen operator and the adjoint of the structure operator, are disjoints. On the other hand, Assumption A3 states a unique continuation property.

We are now in position to state the main results of this chapter.

Theorem 5.2.1. *Let $\alpha \in (0, \alpha^*)$. Let us suppose that Assumptions A1, A2 and A3 are satisfied. For all $\omega > 0$, there exists a family $(w_i)_{i=1}^{N_c} \subset H^2_{\{0\}}(0, \ell_s)$ and an operator*

$$\mathcal{K} \in \mathcal{L}(\mathbf{L}^2(\Omega) \times H^2_{\{0\}}(0, \ell_s) \times L^2(0, \ell_s), \mathbb{R}^{N_c}),$$

for which there exist $R > 0$ and $r > 0$, such that for all $(\widehat{\mathbf{u}}^0, \eta_2^0) \in \mathbf{H}^1(\Omega) \times H^2_{\{0\}}(0, \ell_s)$ and all $\widehat{\mathbf{g}}_p \in H^1_{\{0\}}(0, \infty; \mathbf{H}(\Gamma_i))$ satisfying

$$\begin{aligned} &\widehat{\mathbf{u}}^0 = \widehat{\mathbf{g}}_p(0, \cdot) \ \ \text{on } \Gamma_i, \quad \widehat{\mathbf{u}}^0 = 0 \ \ \text{on } \Gamma_r \cup \Gamma_w, \\ &\widehat{\mathbf{u}}^0 = \eta_2^0(0, \cdot)\vec{\mathbf{e}}_2 \ \ \text{on } \Gamma_s, \quad \operatorname{div} \widehat{\mathbf{u}}^0 = 0 \ \ \text{in } \Omega, \end{aligned} \tag{5.2.18}$$

and

$$\|\widehat{\mathbf{u}}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^2_{\{0\}}(0,\ell_s)} + \|\widehat{\mathbf{g}}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))} \le r, \tag{5.2.19}$$

system (5.2.11) with a feedback control

$$\widehat{f} = \sum_{i=1}^{N_c} \mathcal{K}_i(\widehat{\mathbf{u}}, \widehat{\eta}, \widehat{\eta}_t) w_i, \quad \text{with } \mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_{N_c}),$$

admits a solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}) \in B_\infty(R, \mathbf{u}_0, \eta_2^0)$ satisfying

$$\|(\widehat{\mathbf{u}}(t,\cdot), \widehat{\eta}(t,\cdot), \widehat{\eta}_t(t,\cdot))\|_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega) \times H^3(0,\ell_s) \times H^1(0,\ell_s)} \le CR \ \ \text{for all } t > 0,$$

where $C > 0$ depends on α and r.

The following result follows from the preceding theorem:

Theorem 5.2.2. *Let $\alpha \in (0, \alpha^*)$. Let us suppose that Assumptions A1, A2 and A3 are satisfied. For all $\omega > 0$, there exists a family $(w_i)_{i=1}^{N_c} \subset H^2_{\{0\}}(0, \ell_s)$ and an operator*

$$(\mathcal{K}_1, \dots, \mathcal{K}_{N_c}) \in \mathcal{L}(\mathbf{L}^2(\Omega) \times H^2_{\{0\}}(0, \ell_s) \times L^2(0, \ell_s), \mathbb{R}^{N_c}),$$

for which there exists $r > 0$ such that for all $(\mathbf{u}^0, \eta_2^0) \in \mathbf{H}^1(\Omega) \times H^2_{\{0\}}(0, \ell_s)$ and all $e^{\omega t}\mathbf{g}_p \in$

$H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))$ *satisfying*

$$\begin{aligned}&\mathbf{u}^0-\mathbf{u}_s=\mathbf{g}_p(0,\cdot)\ \ \textit{on}\ \Gamma_i,\ \ \mathbf{u}^0=0\ \ \textit{on}\ \Gamma_r\cup\Gamma_w,\\&\mathbf{u}^0-\mathbf{u}_s=\eta_2^0(0,\cdot)\vec{\mathbf{e}}_2\ \ \textit{on}\ \Gamma_s,\ \ \operatorname{div}\mathbf{u}^0=0\ \ \textit{in}\ \Omega,\end{aligned}\tag{5.2.20}$$

and

$$\|\mathbf{u}^0-\mathbf{u}_s\|_{\mathbf{H}^1(\Omega)}+\|\eta_2^0\|_{H^2(0,\ell_s)}+\|e^{\omega t}\mathbf{g}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))}\le r,\tag{5.2.21}$$

system (5.1.1) *with a feedback control*

$$f=\sum_{i=1}^{N_c}\mathcal{K}_i(\mathbf{u}\circ X^{-1}-\mathbf{u}_s,\eta,\eta_t)w_i,\quad \textit{with}\ \mathcal{K}=(\mathcal{K}_1,\dots,\mathcal{K}_{N_c}),$$

admits a solution $(\mathbf{u},p,\eta)$ *belonging to the class* (5.2.14) *satisfying*

$$\left\|\Big(\mathbf{u}(t,X^{-1}(t,\cdot))-\mathbf{u}_s,\eta(t,\cdot),\eta_t(t,\cdot)\Big)\right\|_{\mathbf{H}^{\frac12+\alpha}(\Omega)\times H^3(0,\ell_s)\times H^1(0,\ell_s)}\le Ce^{-\omega t}\quad \textit{for all}\ t>0,$$

where $C>0$ *depends on* α *and* r.

5.3 Analysis of the fluid-structure operator

In this section we study the linearized stationary fluid-structure system

$$\begin{cases}\lambda\mathbf{v}-\operatorname{div}\sigma(\mathbf{v},q)+(\mathbf{u}_s\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{u}_s-A_1\eta_1-A_2\eta_2=\mathbf{F}_f\ \ \text{in}\ \Omega,\\ \operatorname{div}\mathbf{v}=A_3\eta_1\ \ \text{in}\ \Omega,\\ \mathbf{v}=\eta_2\vec{\mathbf{e}}_2\ \ \text{on}\ \Gamma_s,\ \ \mathbf{v}=0\ \ \text{on}\ \Gamma_d\setminus\Gamma_s,\ \ \sigma(\mathbf{v},q)\mathbf{n}=0\ \ \text{on}\ \Gamma_n,\\ \lambda\eta_1-\eta_2=F_s^1\ \ \text{in}\ (0,\ell_s),\\ \lambda\eta_2+\alpha\Delta^2\eta_1+\gamma(\Delta_s^2)^{\frac12}\eta_2-A_4\eta_1=-\gamma_s^+q+\gamma_s^-q+F_s^2\ \ \text{in}\ (0,\ell_s),\\ \eta_1=0,\partial_{x_1}\eta_1=0\ \ \text{on}\ \{0\}\ \ \text{and}\ \ \partial_{x_1}^2\eta_1=0,\partial_{x_1}^3\eta_1=0\ \ \text{on}\ \{\ell_s\}.\end{cases}\tag{5.3.1}$$

Here, either $\lambda\in\mathbb{C}$ or $\lambda\in\mathbb{R}$. The context will indicate in which case we shall be.

We choose $\lambda_f>0$ large enough to guarantee the following coercivity condition:

$$\lambda_f\int_\Omega|\mathbf{v}|^2+2\nu\int_\Omega|\varepsilon(\mathbf{v})|^2+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{u}_s)\cdot\mathbf{v}\ge\frac{\nu}{2}\|\mathbf{v}\|^2_{\mathbf{H}^1_{\Gamma_d}(\Omega)},\tag{5.3.2}$$

for all $\mathbf{v}\in\mathbf{V}^1_{\Gamma_d}(\Omega)$.

This section is organized as follows. First, in Subsection 5.3.1, we study the fluid system. In Subsection 5.3.2, we introduce the structure operator. Finally, the fluid-structure operator is introduced in Subsection 5.3.3.

5.3.1 Fluid operator

We consider the following fluid system:

$$\begin{cases}\lambda\mathbf{w}-\operatorname{div}\sigma(\mathbf{w},\pi)+(\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s=\mathbf{F}\ \ \text{in}\ \Omega,\\ \operatorname{div}\mathbf{w}=h\ \ \text{in}\ \Omega,\\ \mathbf{w}=\mathbf{g}\ \ \text{on}\ \Gamma_d,\ \ \sigma(\mathbf{w},\pi)\mathbf{n}=0\ \ \text{on}\ \Gamma_n.\end{cases}\tag{5.3.3}$$

Let us assume that $\mathbf{F}\in\mathbf{H}^{-1}_{\Gamma_d}(\Omega)$, $h\in L^2(\Omega)$ and $\mathbf{g}\in\mathbf{H}^{\frac12}(\Gamma_d)$. We will say that the pair $(\mathbf{w},\pi)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ is a variational solution of the system (5.3.3), if and only if it satisfies the following mixed variational formulation:

$$\begin{cases} a(\mathbf{w}, \boldsymbol{\phi}) - b(\boldsymbol{\phi}, \pi) = \langle \mathbf{F}, \boldsymbol{\phi} \rangle_{\mathbf{H}^{-1}_{\Gamma_d}, \mathbf{H}^1_{\Gamma_d}} \ \text{ for all } \boldsymbol{\phi} \in \mathbf{H}^1_{\Gamma_d}(\Omega), \\ b(\mathbf{w}, \psi) = \displaystyle\int_\Omega h\psi \ \text{ for all } \psi \in L^2(\Omega), \\ \mathbf{w} = \mathbf{g} \ \text{ on } \Gamma_d, \end{cases} \tag{5.3.4}$$

where

$$a(\mathbf{w}, \boldsymbol{\phi}) = \int_\Omega (\lambda \mathbf{w} \cdot \boldsymbol{\phi} + 2\nu \varepsilon(\mathbf{w}) : \varepsilon(\boldsymbol{\phi})) + \int_\Omega ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \cdot \boldsymbol{\phi}$$
$$b(\mathbf{w}, \psi) = \int_\Omega (\operatorname{div} \mathbf{w})\, \psi.$$

• **Regularity result of the Oseen system** (5.3.3). The following result is a consequence of the coercivity condition (5.3.2) and Theorems 2.3.1 and 2.3.2 of Chapter 2.

Theorem 5.3.1. *Let $\alpha \in (0, \alpha^*)$, $\delta \in (\delta^*, 1)$ and $\lambda \geq \lambda_f$. Under the Assumption (5.3.2), the following assertions hold:*

(*i*) *For all $(\mathbf{F}, h, \mathbf{g}) \in \mathbf{H}^{-1}_{\Gamma_d}(\Omega) \times L^2(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\Gamma_d)$, system (5.3.3) admits a unique variational solution $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$.*

(*ii*) *For all $(\mathbf{F}, h, \mathbf{g}) \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega) \times H^{\frac{1}{2}+\alpha,1}(\Omega) \times \mathbf{H}^{\frac{3}{2}}(\Gamma_d)$, the variational solution $(\mathbf{w}, \pi)$ of system (5.3.3) belongs to $\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)$. Moreover, there exists a constant $C_\alpha > 0$, such that*

$$\|\mathbf{w}\|_{\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)} + \|\pi\|_{H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)} \leq C_\alpha \Big(\|\mathbf{F}\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} + \|h\|_{H^{\frac{1}{2}+\alpha,1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_d)} \Big). \tag{5.3.5}$$

• **Oseen operator**. Let $\alpha \in (0, \alpha^*)$. Let us first recall that the Leray projector $P \in \mathcal{L}(\mathbf{L}^2(\Omega))$ can be continuously extended from $\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)$ into $\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)$, and from $\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$ into $\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$ (see Subsection 2.4.1 of Chapter 2). For the precise definition of the operator P, see Proposition 2.4.1 in Chapter 2.

The Oseen operator $(A, \mathcal{D}(A))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ that we will consider is defined by

$$\begin{aligned} \mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)) = \Big\{ & \mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \cap \mathbf{H}^1_{\Gamma_d}(\Omega) \mid \exists \pi \in H^{\frac{1}{2}+\alpha}(\Omega) \text{ such that } \operatorname{div} \sigma(\mathbf{w}, \pi) \in \mathbf{L}^2(\Omega) \\ & \text{and } \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \text{ on } \Gamma_n \Big\}, \\ & A\mathbf{w} = P \operatorname{div} \sigma(\mathbf{w}, \pi) - P\left((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s\right). \end{aligned} \tag{5.3.6}$$

The analyticity of the semigroup generated by the Oseen operator $(A, \mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is proved in [NR15, Theorem 2.8].

Using Lemma 3.3.1 presented in Chapter 3 and the fact A is an isomorphism from $\mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega))$ into $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and from $\mathbf{V}^1_{\Gamma_d}(\Omega)$ into $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$ (this follows from the Lax-Milgram theorem), we deduce that A is also an isomorphism from

$$\mathcal{D}(A; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)) = [\mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathbf{V}^1_{\Gamma_d}(\Omega)]_{\frac{1}{2}-\alpha} \text{ into } [\mathbf{V}^0_{n,\Gamma_d}(\Omega), \mathbf{V}^{-1}_{\Gamma_d}(\Omega)]_{\frac{1}{2}-\alpha} = \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega). \tag{5.3.7}$$

In the following theorem, we establish the analyticity of the semigroup generated by the Oseen operator $(A, \mathcal{D}(A; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ on $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.

Theorem 5.3.2. *Let $\alpha \in (0, \alpha^*)$. There exists $\theta_0 \in (\pi/2, \pi)$ and $C > 0$ such that*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega))} \leq \frac{C}{|\lambda|}, \quad \textit{for all } \lambda \in \Sigma_{\theta_0} \setminus \{0\}. \tag{5.3.8}$$

In particular, the unbounded operator $(A, \mathcal{D}(A; \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$.

The proof of Theorem 5.3.2 is presented in Appendix B at the end of this chapter.

- **Expression of the pressure and operator equation**. Let us consider the system

$$\begin{cases} \lambda \mathbf{w} - \operatorname{div} \sigma(\mathbf{w}, \pi) + (\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s = \mathbf{F} \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{w} = h \ \text{ in } \Omega, \\ \mathbf{w} = \mathbf{g}_p \ \text{ on } \Gamma_i, \quad \mathbf{w} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{w} = 0 \ \text{ on } \Gamma_d \setminus (\Gamma_i \cup \Gamma_s), \\ \sigma(\mathbf{w}, \pi)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{5.3.9}$$

The objective is to rewrite system (5.3.9) as an operator equation (see Theorem 5.3.3). To that end, we begin by introducing a family of suitable operators that we will allow us to rewrite the pressure π in system (5.3.9).

Let us first notice that, at least formally, the pressure π in system (5.3.9) is the solution of the system

$$\begin{cases} \Delta \pi = -\lambda h + 2\nu \operatorname{div}(\operatorname{div} \varepsilon(\mathbf{w})) - \operatorname{div}((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) + \operatorname{div} \mathbf{F} \ \text{ in } \Omega, \\ \dfrac{\partial \pi}{\partial \mathbf{n}} = (-\lambda \mathbf{w} + 2\nu \operatorname{div} \varepsilon(\mathbf{w}) - ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s + \mathbf{F}) \cdot \mathbf{n} \text{ on } \Gamma_d, \\ \pi = 2\nu \varepsilon(\mathbf{w})\mathbf{n} \cdot \mathbf{n} \text{ on } \Gamma_n. \end{cases} \tag{5.3.10}$$

We write $\pi = q_1 + q_2 + q_3 + q_4$, where q_i, with $i = 1, 2, 3, 4$, satisfies

$$\Delta q_1 = \operatorname{div} \mathbf{F} \ \text{ in } \Omega, \quad \frac{\partial q_1}{\partial \mathbf{n}} = \mathbf{F} \cdot \mathbf{n} \ \text{ on } \Gamma_d, \quad q_1 = 0 \ \text{ on } \Gamma_n, \tag{5.3.11}$$

$$\Delta q_2 = -\lambda h \ \text{ in } \Omega, \quad \frac{\partial q_2}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \quad q_2 = 0 \ \text{ on } \Gamma_n, \tag{5.3.12}$$

$$\begin{cases} \Delta q_3 = 2\nu \operatorname{div}(\operatorname{div} \varepsilon(\mathbf{w})) - \operatorname{div}((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \ \text{ in } \Omega, \\ \dfrac{\partial q_3}{\partial \mathbf{n}} = 2\nu \operatorname{div} \varepsilon(\mathbf{w}) \cdot \mathbf{n} - ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \cdot \mathbf{n} \ \text{ on } \Gamma_d, \\ q_3 = 2\nu \varepsilon(\mathbf{w})\mathbf{n} \cdot \mathbf{n} \ \text{ on } \Gamma_n, \end{cases} \tag{5.3.13}$$

and

$$\Delta q_4 = 0 \ \text{ in } \Omega, \quad \frac{\partial q_4}{\partial \mathbf{n}} = -\lambda \mathbf{w} \cdot \mathbf{n} \ \text{ on } \Gamma_d, \quad q_4 = 0 \ \text{ on } \Gamma_n. \tag{5.3.14}$$

Operator N_p

We introduce the operator $N_p \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega), H^{\frac{1}{2}+\alpha,1}(\Omega))$, defined by

$$N_p \mathbf{F} = q_1, \tag{5.3.15}$$

where q_1 solves system (5.3.11) in the sense of (2.4.21) introduced in Chapter 2. Let us notice that the operator N_p is well-defined thanks to Lemma 2.4.4.

Operator $N_{\rm div}$

Given $h \in L^2(\Omega)$, let us consider the elliptic equation

$$\Delta q = h \ \text{ in } \Omega, \quad \frac{\partial q}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \quad q = 0 \ \text{ on } \Gamma_n. \tag{5.3.16}$$

We introduce the operator $N_{\rm div}$, defined by

$$N_{\rm div} \in \mathcal{L}(L^2(\Omega), H^1(\Omega)), \ N_{\rm div} h = q, \tag{5.3.17}$$

where q is the variational solution of system (5.3.16).

Operator N_v

We first define the variational problem satisfied by q_3:

$$\begin{aligned}
&\text{Find } q_3 \in L^2(\Omega) \text{ such that} \\
&\int_\Omega q_3 \zeta \, dx = 2\nu \langle \varepsilon(\mathbf{w}), \nabla^2 \chi \rangle_{\mathbf{H}^{\frac12 - \alpha}, \mathbf{H}^{-\frac12+\alpha}} - 2\nu \int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n} \cdot \nabla \chi \\
&\qquad + \int_\Omega ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \cdot \nabla \chi,
\end{aligned} \tag{5.3.18}$$

for all $\zeta \in L^2(\Omega)$, where χ is solution to

$$\Delta \chi = \zeta \ \text{ in } \Omega, \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \quad \chi = 0 \ \text{ on } \Gamma_n. \tag{5.3.19}$$

Lemma 5.3.1. *The variational problem (5.3.18) admits a unique solution $q_3 \in L^2(\Omega)$.*

Proof. Let us introduce the linear funcional $L : L^2(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}
L(\zeta) = {} & 2\nu \langle \varepsilon(\mathbf{w}), \nabla^2 \chi \rangle_{\mathbf{H}^{\frac12-\alpha}_{\Gamma_d}(\Omega), \mathbf{H}^{-\frac12+\alpha}_{\Gamma_d}(\Omega)} - 2\nu \int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n} \cdot \nabla \chi \\
& + \int_\Omega ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \cdot \nabla \chi
\end{aligned}$$

where χ is solution to (5.3.19) with source term $\zeta \in L^2(\Omega)$. Let us notice that

$$\begin{aligned}
2\nu \langle \varepsilon(\mathbf{w}), \nabla^2 \psi \rangle_{\mathbf{H}^{\frac12-\alpha}(\Omega), \mathbf{H}^{-\frac12+\alpha}(\Omega)} &\le C \|\varepsilon(\mathbf{w})\|_{\mathbf{H}^{\frac12-\alpha}(\Omega)} \|\nabla^2 \psi\|_{\mathbf{H}^{-\frac12+\alpha}(\Omega)} \\
&\le C \|\mathbf{w}\|_{H^{\frac32+\alpha}(\Omega)} \|\zeta\|_{L^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
-2\nu \int_{\Gamma_d} \varepsilon(\mathbf{w})\mathbf{n} \cdot \nabla \psi &\le C \|\varepsilon(\mathbf{w})\|_{\mathbf{L}^2(\Gamma_d)} \|\nabla \psi\|_{\mathbf{L}^2(\Gamma_d)} \\
&\le C \|\mathbf{w}\|_{\mathbf{H}^{\frac32+\alpha}(\Omega)} \|\zeta\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega ((\mathbf{u}_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}_s) \cdot \nabla \chi &\le C \|\mathbf{w}\|_{\mathbf{H}^{\frac32+\alpha}(\Omega)} \|\nabla \chi\|_{\mathbf{L}^2(\Omega)} \\
&\le C \|\mathbf{w}\|_{\mathbf{H}^{\frac32+\alpha}(\Omega)} \|\zeta\|_{L^2(\Omega)}.
\end{aligned}$$

Since $L \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, the conclusion follows from the Riesz Representation Theorem. $\square$

We now introduce the operator N_v:

$$N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega), L^2(\Omega)), \quad N_v \mathbf{w} = q_3, \tag{5.3.20}$$

where q_3 is the solution to system (5.3.18).

Operator N_s

For a given $\eta_2 \in L^2(0, \ell_s)$, we consider the elliptic equation

$$\begin{cases} \Delta q = 0 \ \text{ in } \Omega, \ \dfrac{\partial q}{\partial \mathbf{n}} = \eta_2 \text{ on } \Gamma_s^+, \ \dfrac{\partial q}{\partial \mathbf{n}} = -\eta_2 \text{ on } \Gamma_s^-, \\ \dfrac{\partial q}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \ q = 0 \text{ on } \Gamma_n. \end{cases} \tag{5.3.21}$$

Notice that the system (5.3.21) admits a unique variational solution $q_3 \in H^1(\Omega)$. Let us now introduce the operator N_s:

$$N_s \in \mathcal{L}(L^2(0, \ell_s), H^1(\Omega)), \ N_s \eta_2 = q, \tag{5.3.22}$$

where q is the variational solution to system (5.3.21).

Before stating the main result of this subsection, we introduce the lifting operators

$$L \in \mathcal{L}(H^2_{\{0\}}(0, \ell_s) \times H^{\frac{1}{2}+\alpha,1}(\Omega), \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)), \quad L(\eta_2, h) = \mathbf{w} \tag{5.3.23}$$

and

$$L_p \in \mathcal{L}(H^2_{\{0\}}(0, \ell_s) \times H^{\frac{1}{2}+\alpha,1}(\Omega), H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)), \quad L_p(\eta_2, h) = \pi, \tag{5.3.24}$$

where $(\mathbf{w}, \pi)$ is the solution to system (5.3.9) when $\mathbf{F} = 0$, $\mathbf{g}_p = 0$ and $\lambda = \lambda_f$.

Theorem 5.3.3. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Assume that $\mathbf{F} \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$, $h \in H^{\frac{1}{2}+\alpha,1}(\Omega)$, $\eta_2 \in H^2_{\{0\}}(0, \ell_s)$ and $\mathbf{g}_p = 0$. A pair $(\mathbf{w}, \pi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)$ is a variational solution of (5.3.9) if and only if $P\mathbf{w}$, $(I-P)\mathbf{w}$, and π are solutions to the system*

$$\begin{cases} (\lambda I - A)P\mathbf{w} + (A - \lambda_f I)PL(\eta_2, h) = P\mathbf{F}, \ \ (I-P)\mathbf{w} = (I-P)L(\eta_2, h), \\ \pi = \lambda N_s \eta_2 - \lambda N_{\mathrm{div}} h + N_v \mathbf{w} + N_p \mathbf{F}. \end{cases} \tag{5.3.25}$$

Proof. Let $(\mathbf{w}, \pi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)$ be the solution of (5.3.9). We set $\mathbf{w} = \widehat{\mathbf{w}} + L(\eta_2, h)$ and $\pi = \widehat{\pi} + L_p(\eta_2, h)$, where the couple $(\widehat{\mathbf{w}}, \widehat{\pi})$ satisfies

$$\begin{cases} \lambda \widehat{\mathbf{w}} - \operatorname{div} \sigma(\widehat{\mathbf{w}}, \widehat{\pi}) + (\mathbf{u}_s \cdot \nabla)\widehat{\mathbf{w}} + (\widehat{\mathbf{w}} \cdot \nabla)\mathbf{u}_s = \mathbf{F} - (\lambda - \lambda_f)L(\eta_2, h) \ \text{ in } \Omega, \\ \operatorname{div} \widehat{\mathbf{w}} = 0 \ \text{ in } \Omega, \\ \widehat{\mathbf{w}} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\widehat{\mathbf{w}}, \pi)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{5.3.26}$$

We notice that the pair $(\widehat{\mathbf{w}}, \widehat{\pi})$ satisfies the system (5.3.9) for $(\eta_2, h) = (0, 0)$ and right-hand side equals to $\mathbf{F} - (\lambda - \lambda_f)L(\eta_2, h)$, and therefore $\widehat{\mathbf{w}} \in \mathcal{D}(A)$. Thus, $\lambda P\widehat{\mathbf{w}} - AP\widehat{\mathbf{w}} = P\mathbf{F} - (\lambda - \lambda_f)PL(\eta_2, h)$. Then, since $\widehat{\mathbf{w}} = \mathbf{w} - L(\eta_2, h)$, we get $(\lambda I - A)P\mathbf{w} + (A - \lambda_f I)PL(\eta_2, h) = P\mathbf{F}$. Furthermore, the algebraic constraint $(I-P)\mathbf{w} = (I-P)L(\eta_2, h)$ holds.

We now proceed to derive the expression of the pressure π. We first show the identity

$$\overbrace{\lambda\int_\Omega \mathbf{w}\cdot\nabla\chi}^{I_1}\overbrace{-\langle \operatorname{div}\sigma(\mathbf{w},\pi),\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi}^{I_2}$$
$$=-\lambda\int_\Omega N_{\mathrm{div}}h\zeta+\lambda\int_\Omega N_s\eta_2\zeta-\int_\Omega\pi\zeta+\int_\Omega N_v\mathbf{w}\zeta. \tag{5.3.27}$$

$\underline{I_1}$: After integrating by parts, we get

$$I_1=\lambda\int_\Omega \mathbf{w}\cdot\nabla\chi=-\lambda\int_\Omega(\operatorname{div}\mathbf{w})\chi+\lambda\int_{\partial\Omega}\mathbf{w}\cdot\mathbf{n}\chi=-\lambda\int_\Omega N_{\mathrm{div}}h\zeta+\lambda\int_\Omega N_s\eta_2\zeta. \tag{5.3.28}$$

$\underline{I_2}$: We will prove that

$$I_2=-\int_\Omega\pi\zeta+\int_\Omega N_v\mathbf{w}\zeta. \tag{5.3.29}$$

Let $(\mathbf{w}_k)_k$ be a sequence in $\mathbf{H}^2(\Omega)$ converging to $\mathbf{w}$ in $\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and let $(\pi_k)_k$ be a sequence in $H^1(\Omega)$ converging to π in $H^{\frac{1}{2}+\alpha}(\Omega)$. Let us first observe that

$$-\langle \operatorname{div}\sigma(\mathbf{w}_k,\pi_k),\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_k\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi$$
$$\xrightarrow[k\to+\infty]{}-\langle \operatorname{div}\sigma(\mathbf{w},\pi),\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi,$$

where χ is the solution of (5.3.19). On the other hand, since $N_v\in\mathcal{L}(\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega),L^2(\Omega))$ and $\sigma(\mathbf{w},\pi)\mathbf{n}=0$ on Γ_n, we have

$$\begin{aligned}
&-\langle \operatorname{div}\sigma(\mathbf{w}_k,\pi_k),\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_k\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi\\
&=-\int_\Omega\operatorname{div}\sigma(\mathbf{w}_k,\pi_k)\cdot\nabla\chi+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_k\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi\\
&=\langle\nabla^2\chi,\sigma(\mathbf{w}_k,\pi_k)\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}-\int_\Gamma\sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi\\
&\qquad\qquad+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_k\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi\\
&=-\int_\Omega\pi_k\Delta\chi+2\nu\langle\nabla^2\chi,\varepsilon(\mathbf{w}_k)\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}\\
&+\int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_k\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi-2\nu\int_{\Gamma_d}\varepsilon(\mathbf{w}_k)\mathbf{n}\cdot\nabla\chi-\int_{\Gamma_n}\sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi\\
&=-\int_\Omega\pi_k\zeta+\int_\Omega N_v\mathbf{w}_k\zeta-\int_{\Gamma_n}\sigma(\mathbf{w}_k,\pi_k)\mathbf{n}\cdot\nabla\chi\\
&\xrightarrow[k\to+\infty]{}-\int_\Omega\pi\zeta+\int_\Omega N_v\mathbf{w}\zeta-\int_{\Gamma_n}\sigma(\mathbf{w},\pi)\mathbf{n}\cdot\nabla\chi=-\int_\Omega\pi\zeta+\int_\Omega N_v\mathbf{w}\zeta.
\end{aligned} \tag{5.3.30}$$

Thus, using (5.3.30) and the uniqueness of the limit, we deduce the identity (5.3.29).

On the other hand, from the definition of the operator N_p (see (5.3.15)), we have that

$$\langle\mathbf{F},\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)}=-\int_\Omega N_p\mathbf{F}\zeta. \tag{5.3.31}$$

Finally, since

$$\begin{aligned}&\lambda\int_\Omega \mathbf{w}\cdot\nabla\chi - \langle \operatorname{div}\sigma(\mathbf{w},\pi),\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)} + \int_\Omega ((\mathbf{u}_s\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}_s)\cdot\nabla\chi\\ &= \langle \mathbf{F},\nabla\chi\rangle_{\mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}_{\Gamma_d}^{\frac{1}{2}-\alpha}(\Omega)},\end{aligned}\tag{5.3.32}$$

the identities (5.3.27) and (5.3.31) allow us to obtain

$$\int_\Omega \pi\zeta = -\lambda\int_\Omega N_{\text{div}}h\zeta + \lambda\int_\Omega N_s\eta_2\zeta + \int_\Omega N_v\mathbf{w}\zeta + \int_\Omega N_p\mathbf{F}\zeta,$$

from where we deduce that $\pi = -\lambda N_{\text{div}}h + \lambda N_s\eta_2 + N_v\mathbf{w} + N_p\mathbf{F}$. This concludes the proof of the first implication.

Let us now prove the converse. Assume that the couple $(\mathbf{w},\pi)\in\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)\times H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ satisfies (5.3.25). We claim that (5.3.25) admits a unique solution. Indeed, let us notice that $((\lambda I - A)\mathbf{w},\mathbf{w})_{\mathbf{L}^2(\Omega)} = 0$ implies that $\mathbf{w}=0$. Next, $(\lambda I - A)P\mathbf{w} = 0$ and $(I-P)\mathbf{w}=0$, leads to $\mathbf{w}=0$, thereby ensuring the uniqueness. This completes the proof. □

Remark 13. Since $\pi = -\lambda N_{\text{div}}h + \lambda N_s\eta_2 + N_v\mathbf{w} + N_p\mathbf{F}$ and $\pi\in H^{\frac{1}{2}+\alpha,1}(\Omega)$, $N_{\text{div}}h\in H^1(\Omega)$, $N_s\eta_2\in H^1(\Omega)$, $N_p\mathbf{F}\in H^{\frac{1}{2}+\alpha,1}(\Omega)$, we deduce that, indeed, $N_v\mathbf{w}\in H^{\frac{1}{2}+\alpha,1}(\Omega)$.

5.3.2 Structure operator

In this subsection, we revisit some results established in Chapter 3 concerning the structure operator (see Subsection 3.3.2).

We introduce the state space

$$H_s = H^2_{\{0\}}(0,\ell_s)\times L^2(0,\ell_s),\tag{5.3.33}$$

which is endowed with the inner product

$$\langle(\eta_1,\eta_2),(\zeta_1,\zeta_2)\rangle_{H_s} = \int_0^{\ell_s}(\alpha\eta_{1,xx}\zeta_{1,xx}+\eta_2\zeta_2)\,dx.$$

We consider the unbounded operator $(\Delta_s^2, D(\Delta_s^2))$ in $L^2(0,\ell_s)$, where

$$D(\Delta_s^2) = H^4_{\{0,\ell_s\}}(0,\ell_s).$$

We now define the unbounded operator $(A_s,\mathcal{D}(A_s))$ in H_s by

$$\mathcal{D}(A_s) = D(\Delta_s^2)\times H^2_{\{0\}}(0,\ell_s),\quad A_s = \begin{pmatrix}0 & I\\ -\alpha\Delta_s^2 & -\gamma(\Delta_s^2)^{\frac{1}{2}}\end{pmatrix}.\tag{5.3.34}$$

Theorem 5.3.4. *The unbounded operator $(A_s,\mathcal{D}(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s.*

We consider the structure equation

$$\begin{cases}\eta_{1,t} = \eta_2 \text{ in } (0,\infty)\times(0,\ell_s),\\ \eta_{2,t}+\alpha\Delta_s^2\eta_1+\gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 = 0 \text{ in } (0,\infty)\times(0,\ell_s),\\ \eta_1 = 0 \text{ and } \eta_{1,x}=0 \text{ on } (0,\infty)\times\{0\},\\ \eta_{1,xx}=0 \text{ and } \eta_{1,xxx}=0 \text{ on } (0,\infty)\times\{\ell_s\},\\ \eta(0)=0 \text{ and } \eta_2(0)=\eta_2^0 \text{ in } (0,\ell_s).\end{cases}\tag{5.3.35}$$

Proposition 5.3.1. *Let us assume that* $\eta_2^0 \in H^1_{\{0\}}(0,\ell_s)$. *Then, system* (5.3.35) *admits a unique solution* $(\eta_1,\eta_2) \in H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s)) \times H^{2,1}((0,\infty)\times(0,\ell_s))$.

5.3.3 Fluid-Structure operator

Let $\alpha \in (0,\alpha^*)$. We introduce the spaces $\mathbf{H} = \mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega) \times H_s$ and $\mathbf{Z} = \mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega) \times H_s$. Both spaces are equipped with the inner product

$$\Big\langle (\mathbf{u},\eta_1,\eta_2),(\mathbf{v},\zeta_1,\zeta_2)\Big\rangle_{\mathbf{Z}} = \Big\langle \mathbf{u},\mathbf{v}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)} + \int_0^{\ell_s} (\alpha\eta_{1,xx}\zeta_{1,xx} + \eta_2\zeta_2)\,dx. \tag{5.3.36}$$

We also introduce the spaces

$$\begin{aligned}\mathbf{Z}_0 &= \mathbf{V}^0_{n,\Gamma_d}(\Omega)\times H_s, \ \ \mathbf{H}_0 = \mathbf{L}^2(\Omega)\times H_s,\\ \mathbf{H}^\alpha_\delta &= \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega) \times H^4_{\{0,\ell_s\}}(0,\ell_s) \times H^2_{\{0\}}(0,\ell_s),\end{aligned}$$

which are equipped with the natural norms.

We begin by introducing some properties of the operators involved in the definition of the fluid-structure operator. We start with the following proposition concerning the operators A_1, A_2, A_3 and A_4.

Proposition 5.3.2. *Let* $\alpha \in (0,\alpha^*)$. *Let* $a_0 \in (0,1/2)$ *be the parameter introduced in Proposition* 5.2.1. *The following assertions hold.*

(*i*) $A_1 \in \mathcal{L}(H^2_{\{0\}}(0,\ell_s), \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$,
(*ii*) $A_2 \in \mathcal{L}(L^2(0,\ell_s), \mathbf{L}^2(\Omega))$,
(*iii*) $A_3 \in \mathcal{L}(H^2_{\{0\}}(0,\ell_s), L^2(\Omega))$
(*iv*) $A_4 \in \mathcal{L}(H^2_{\{0\}}(0,\ell_s), L^2(0,\ell_s))$.

Proof. The first assertion follows from [GS91, Proposition B1] and Lemma 3.5.3 of Chapter 3. The assertions (*ii*), (*iii*) and (*iv*) follow from [GS91, Proposition B1]. □

Let us now introduce the so-called *added mass operator* $M \in \mathcal{L}(\mathbf{Z}_0)$, which is defined by

$$M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\gamma_s^{+,-}N_{\mathrm{div}}A_3 & I+\gamma_s^{+,-}N_s \end{pmatrix}, \tag{5.3.37}$$

where $\gamma_s^{+,-} = \gamma_s^+ - \gamma_s^-$. Thanks to Lemma 3.3.2, we have that $M^{-1} \in \mathcal{L}(\mathbf{Z}_0)$. Moreover,

$$M^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & (I+\gamma_s^{+,-}N_s)^{-1}\gamma_s^{+,-}N_{\mathrm{div}}A_3 & (I+\gamma_s^{+,-}N_s)^{-1} \end{pmatrix}. \tag{5.3.38}$$

We now introduce the fluid-structure operator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$:

$$\begin{aligned}\mathcal{A} &= M^{-1}\begin{pmatrix} A & PA_1 & PA_2 + (\lambda_f I - A)PL(\cdot,0) \\ 0 & 0 & I \\ -\gamma_s^{+,-}N_v & -\alpha\Delta_s^2 + A_4 & -\gamma B - \gamma_s^{+,-}N_pA_2 \end{pmatrix} + M^{-1}A_p,\\ \mathcal{D}(\mathcal{A}) &= \Big\{ (P\mathbf{v},\eta_1,\eta_2) \in \mathbf{V}^{\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)\times\mathcal{D}(A_s) \mid P\mathbf{v} - PL(\eta_2, A_3\eta_1) \in \mathcal{D}(A)\Big\},\end{aligned} \tag{5.3.39}$$

where

$$A_p\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (\lambda_f I - A)PL(0,A_3\eta_1) \\ -\gamma_s^{+,-}N_pA_1\eta_1 \\ -\gamma_s^{+,-}N_v\nabla N_{\mathrm{div}}A_3\eta_1 + \gamma_s^{+,-}N_v\nabla N_s\eta_2 \end{pmatrix}.$$

In the next result, we provide an operator-based formulation of system (5.3.1), which follows as a consequence of Theorem 5.3.3 and the fact that

$$\nabla N_p = I - P, \;\; (I-P)L(0, A_3\eta_1) = \nabla N_{\rm div} A_3\eta_1, \;\; (I-P)L(\eta_2, 0) = -\nabla N_s\eta_2.$$

Theorem 5.3.5. *Let us assume that* $(\mathbf{F}_f, F_s^1, F_s^2) \in \mathbf{H}_{\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega) \times H_s$ *and* $\lambda \in \mathbb{C}$. *Then, the quadruplet* $(\mathbf{v}, q, \eta_1, \eta_2)$ *is solution to system* (5.3.1) *if and only if,*

$$\begin{cases} \lambda(P\mathbf{v}, \eta_1, \eta_2)^\top = \mathcal{A}(P\mathbf{v}, \eta_1, \eta_2)^\top + M^{-1}(P\mathbf{F}_f, F_s^1, F_s^2 - \gamma_s^{+,-}N_p\mathbf{F}_f), \\ (I-P)\mathbf{v} = \nabla N_{\rm div} A_3\eta_1 - \nabla N_s\eta_2, \\ q = -\lambda N_{\rm div}A_3\eta_1 + \lambda N_s\eta_2 + N_pA_1\eta_1 + N_pA_2\eta_2 + N_v(P\mathbf{v} - \nabla N_s\eta_2 + \nabla N_{\rm div}A_3\eta_1). \end{cases} \tag{5.3.40}$$

5.4 Characterization of the adjoint operator $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$

We remark that in this section the adjoint fluid-structure operator is obtained only through formal computations. The justification of the well-posedness of the adjoint system involved in these computations remains to be completed.

Let $\alpha \in (0, \alpha^*)$. Given $\mathbf{G}_f \in \mathbf{L}^2(\Omega)$ and $\zeta_2 \in H^2_{\{0\}}(0, \ell_s)$, let us first consider the system

$$\begin{cases} \lambda\mathbf{\Phi} - {\rm div}\,\sigma(\mathbf{\Phi}, \psi) - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi} + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi} = \mathbf{G}_f \;\text{ in } \Omega, \\ {\rm div}\,\mathbf{\Phi} = 0 \;\text{ in } \Omega, \\ \mathbf{\Phi} = \zeta_2\vec{\mathbf{e}}_2 \;\text{ on } \Gamma_s, \;\; \mathbf{\Phi} = 0 \;\text{ on } \Gamma_d\setminus\Gamma_s, \;\; \sigma(\mathbf{\Phi}, \psi)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi} = 0 \;\text{ on } \Gamma_n. \end{cases} \tag{5.4.1}$$

The result below follows from [NR15, Theorem 2.11].

Theorem 5.4.1. *The adjoint operator of* $(A, \mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ *is defined by*

$$\begin{aligned} \mathcal{D}(A^*) = \Big\{&\mathbf{\Phi} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \cap \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists\psi \in H^{\frac{1}{2}+\alpha}(\Omega) \textit{ such that} \\ &{\rm div}\,\sigma(\mathbf{\Phi}, \psi) \in \mathbf{L}^2(\Omega) \;\textit{ and }\; \sigma(\mathbf{\Phi}, \psi)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi} = 0 \textit{ on } \Gamma_n\Big\}, \end{aligned} \tag{5.4.2}$$

$$A^*\mathbf{\Phi} = P\,{\rm div}\,\sigma(\mathbf{\Phi}, \psi) - P\left((\nabla\mathbf{u}_s)^\top\mathbf{\Phi} - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi}\right).$$

Let us now introduce the lifting operators $D \in \mathcal{L}(H^2_{\{0\}}(0, \ell_s), \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))$ and $D_p \in \mathcal{L}(H^2_{\{0\}}(0, \ell_s), H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$ for the adjoint system, defined by

$$D\zeta_2 = \mathbf{\Phi} \;\text{ and }\; D_p\zeta_2 = \psi, \tag{5.4.3}$$

where the couple $(\mathbf{\Phi}, \psi)$ is the solution of the system (5.4.1) with $\lambda = \lambda_f$ and $\mathbf{G}_f = 0$.

In order to characterize the pressure ψ in (5.4.1), we also introduce the operator $N_\Phi \in \mathcal{L}(\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega), L^2(\Omega))$, which is defined by $N_\Phi\mathbf{\Phi} = q$, where q is such that

$$\begin{aligned} \int_\Omega q\zeta = {}& 2\nu\Big\langle\varepsilon(\mathbf{\Phi}), \nabla^2\chi\Big\rangle_{\mathbf{H}^{\frac{1}{2}-\alpha}_{\Gamma_d}(\Omega), \mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega)} - 2\nu\int_{\Gamma_d}\varepsilon(\mathbf{\Phi})\mathbf{n}\cdot\nabla\chi \\ &+ \int_\Omega\Big(-(\mathbf{u}_s\cdot\nabla)\mathbf{w} + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi}\Big)\cdot\nabla\chi + \int_{\Gamma_n}(\mathbf{u}_s\cdot\mathbf{n})(\mathbf{\Phi}\cdot\nabla\chi), \end{aligned} \tag{5.4.4}$$

for all $\zeta \in L^2(\Omega)$, where χ is the solution of the system

$$\Delta\chi = \zeta \ \text{ in } \Omega, \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \quad \chi = 0 \ \text{ on } \Gamma_n. \tag{5.4.5}$$

The following result is a consequence of Theorem 5.3.3.

Theorem 5.4.2. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Assume that $\mathbf{G}_f \in \mathbf{L}^2(\Omega)$ and $\zeta_2 \in H^2_{\{0\}}(0, \ell_s)$. Then, the pair $(\mathbf{\Phi}, \psi) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)$ is the solution of (5.4.1), if and only if, the triple $(P\mathbf{\Phi}, \zeta_1, \zeta_2)$ satisfies*

$$\begin{cases} (\lambda I - A^*)\, P\mathbf{\Phi} + (A^* - \lambda_f I) P D \zeta_2 = P\mathbf{G}_f, \\ (I - P)\mathbf{\Phi} = -\nabla N_s \zeta_2, \\ \psi = \lambda N_s \zeta_2 + N_\Phi \left(P\mathbf{\Phi} - \nabla N_s \zeta_2\right) + N_p \mathbf{G}_f. \end{cases} \tag{5.4.6}$$

Let us now consider the adjoint fluid-structure equation

$$\begin{cases} \lambda\mathbf{\Phi} - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) - (\mathbf{u}_s \cdot \nabla)\mathbf{\Phi} + (\nabla \mathbf{u}_s)^\top \mathbf{\Phi} = \mathbf{G}_f \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathbf{\Phi} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d \setminus \Gamma_s, \quad \sigma(\mathbf{\Phi}, \psi)\mathbf{n} + \mathbf{u}_s \cdot \mathbf{n}\mathbf{\Phi} = 0 \ \text{ on } \Gamma_n, \\ \lambda\zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm} A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1} A_1^* \mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1} A_3^* \psi = G_s^1 \ \text{ in } (0, \ell_s), \\ \lambda\zeta_2 + \alpha\Delta_s^2 \zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^* \mathbf{\Phi} = -\gamma_s^+ \psi + \gamma_s^- \psi + G_s^2 \ \text{ in } (0, \ell_s), \\ \zeta_1(0) = \partial_x \zeta_1(0) = 0 \ \text{ and } \ \partial_x^2 \zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_2(\ell_s), \ \ \partial_x^3 \zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s). \end{cases} \tag{5.4.7}$$

Remark 14. We warn the reader that the well-posedness of system (5.4.7) has not yet been established. Thus, all the computations presented below are purely formal.

Proposition 5.4.1. *The adjoint of the added mass operator $M \in \mathcal{L}(\mathbf{Z})$ introduced in (5.3.37) is given by*

$$M^* = \begin{pmatrix} I & 0 & 0 \\ 0 & I & \frac{1}{\alpha}(\Delta_s)^{-1} A_3^* N_s \\ 0 & 0 & I + \gamma_s^{+,-} N_s \end{pmatrix}.$$

Furthermore,

$$(M^*)^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\frac{1}{\alpha}(\Delta_s)^{-1} A_3^* N_s (I + \gamma_s^{+,-} N_s)^{-1} \\ 0 & 0 & (I + \gamma_s^{+,-} N_s)^{-1} \end{pmatrix}.$$

Proof. Let us first observe that the operator $I + \gamma_s^{+,-} N_s \in \mathcal{L}(L^2(0, \ell_s))$ is symmetric. On the other hand,

$$\begin{aligned} -\int_{\Gamma_s} \gamma_s^{+,-}(N_{\text{div}}[A_3\eta_1])\zeta_2 &= -\int_{\Gamma_s} (N_{\text{div}}[A_3\eta_1]) \frac{\partial N_s \zeta_2}{\partial \mathbf{n}} \\ &= -\int_\Omega (\Delta N_s \zeta_2)(N_{\text{div}}[A_3\eta_1]) - \int_\Omega (\nabla N_s \zeta_2) \cdot (\nabla N_{\text{div}}[A_3\eta_1]) \\ &= \int_\Omega (N_s \zeta_2)(\Delta N_{\text{div}}[A_3\eta_1]) - \int_{\partial\Omega} (N_s \zeta_2) \frac{\partial N_{\text{div}} A_3 \eta_1}{\partial \mathbf{n}} \\ &= \int_\Omega (N_s \zeta_2)(A_3 \eta_1) \\ &= \Big(\eta_1, (1/\alpha)(\Delta_s)^{-1} A_3^* N_s \zeta_2\Big)_{L^2(0,\ell_s)}. \end{aligned}$$

Thus, for all $(\mathbf{v}, \eta_1, \eta_2)$ and $(\mathbf{\Phi}, \zeta_1, \zeta_2)$ in $\mathbf{Z}$, we obtain

$$\left\langle M(\mathbf{v}, \eta_1, \eta_2)^\top, (\mathbf{\Phi}, \zeta_1, \zeta_2)^\top \right\rangle_{\mathbf{Z}',\mathbf{Z}} = \left\langle (\mathbf{v}, \eta_1, \eta_2)^\top, M^*(\mathbf{\Phi}, \zeta_1, \zeta_2)^\top \right\rangle_{\mathbf{Z}',\mathbf{Z}}.$$

This conclude the proof of the first part. The second part follows from Lemma 3.3.2 stated in Chapter 3. □

We introduce the space $E_s(0, \ell_s)$ defined by

$$E_s(0,\ell_s) = \Big\{(\zeta_1,\zeta_2) \in (H^4(0,\ell_s) \cap H^2_{\{0\}}(0,\ell_s)) \times H^2_{\{0\}}(0,\ell_s) \mid {\zeta_1}''(\ell_s) = \frac{2\nu e}{\alpha}\zeta_2(\ell_s) \\ \text{and } {\zeta_1}'''(\ell_s) = \frac{2\nu e}{\alpha}\zeta_2'(\ell_s)\Big\}.$$

Before presenting the characterization of the adjoint of the fluid-structure operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $\mathbf{Z}$, we introduce the unbounded operator $(\mathcal{A}^\#, \mathcal{D}(\mathcal{A}^\#))$ on $\mathbf{Z}$ defined by

$$\mathcal{D}(\mathcal{A}^\#) = \Big\{(P\mathbf{\Phi}, \zeta_1, \zeta_2) \in \mathbf{V}^{\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega) \times E_s(0,\ell_s) \mid P(\mathbf{\Phi} - D\zeta_2) \in \mathcal{D}(A^*)\Big\},$$

$$\mathcal{A}^\# = \begin{pmatrix} A^* & 0 & (\lambda_f I - A^*)PD \\ \frac{1}{\alpha}(\Delta_s)^{-1}A_1^* & 0 & -I + \frac{1}{\alpha}(\Delta_s)^{-1}\left(A_4^* - A_1^*\nabla N_s\right) \\ A_2^* - \gamma_s^{+,-}N_\Phi & \alpha\Delta_s^2 & -\delta(\Delta_s^2)^{\frac{1}{2}} - A_2^*\nabla N_s + \gamma_s^{+,-}N_\Phi\nabla N_s \end{pmatrix} + \mathcal{A}_p^\#,$$

where

$$\mathcal{A}_p^\# \begin{pmatrix} P\mathbf{\Phi} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\alpha}(\Delta_s)^{-1}A_3^* N_\Phi\left(P\mathbf{\Phi} - \nabla N_s\zeta_2\right) \\ 0 \end{pmatrix}.$$

Theorem 5.4.3. *Let $\alpha \in (0, \alpha^*)$ and $\delta \in (\delta^*, 1)$. Let us assume that $\lambda \in \mathbb{C}$ and $(\mathbf{G}_f, G_s^1, G_s^2) \in \mathbf{H}^{-\frac{1}{2}+\alpha,0}_{\Gamma_d}(\Omega) \times H_s$. A quadruplet $(\mathbf{\Phi}, \psi, \zeta_1, \zeta_2) \in \mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^{\frac{1}{2}+\alpha,1}_\delta(\Omega) \times E_s(0,\ell_s)$ is a solution of system (5.4.7) if and only if,*

$$\left\{\begin{aligned} &\lambda M^* \begin{pmatrix} P\mathbf{\Phi} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \mathcal{A}^\# \begin{pmatrix} P\mathbf{\Phi} \\ \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{G}_f \\ G_s^1 - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^* N_p\mathbf{G}_f \\ G_s^2 - \gamma_s^{+,-}N_p\mathbf{G}_f \end{pmatrix}, \\ &(I-P)\mathbf{\Phi} = -\nabla N_s\zeta_2, \\ &\psi = \lambda N_s\zeta_2 + N_\Phi\left(P\mathbf{\Phi} - \nabla N_s\zeta_2\right) + N_p\mathbf{G}_f. \end{aligned}\right. \tag{5.4.8}$$

Proof. From Theorem 5.4.2 it follows that the system (5.4.1) is equivalent to the system

$$\left\{\begin{aligned} &\lambda P\mathbf{\Phi} = A^*P\mathbf{\Phi} + \left(\lambda_f I - A^*\right)PD\zeta_2 + P\mathbf{G}_f \\ &(I-P)\mathbf{\Phi} = -\nabla N_s\zeta_2, \\ &\psi = \lambda N_s\zeta_2 + N_\Phi\left(P\mathbf{\Phi} - \nabla N_s D\zeta_2\right) + N_p\mathbf{G}_f. \end{aligned}\right. \tag{5.4.9}$$

Now, we will use the expression for the pressure ψ given in (5.4.9) to rewrite the fourth and fifth equations in system (5.4.7). The fourth equation in system (5.4.7) can be rewritten as

$$\begin{aligned}
\lambda\zeta_1 =& -\zeta_2 + \frac{1}{\alpha}(\Delta_s)^{-1}A_4^*\zeta_2 + \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*P\mathbf{\Phi} - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\nabla N_s\zeta_2 \\
& -\lambda\frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_s\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_\Phi(P\mathbf{\Phi} - \nabla N_s\zeta_2) - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p\mathbf{G}_f + G_s^1 \\
=& \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*P\mathbf{\Phi} + \left[-I + \frac{1}{\alpha}(\Delta_s)^{-1}\left(A_4^* - A_1^*\nabla N_s\right)\right]\zeta_2 \\
& - \frac{\lambda}{\alpha}(\Delta_s)^{-1}A_3^*N_s\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_\Phi\left[P\mathbf{\Phi} - \nabla N_s\zeta_2\right] - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p\mathbf{G}_f + G_s^1,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\lambda\zeta_1 + \lambda\frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_s\zeta_2 =& \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*P\mathbf{\Phi} + \left[-I + \frac{1}{\alpha}(\Delta_s)^{-1}\left(A_4^* - A_1^*\nabla N_s\right)\right]\zeta_2 \\
& - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_\Phi\left[P\mathbf{\Phi} - \nabla N_s\zeta_2\right] - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p\mathbf{G}_f + G_s^1.
\end{aligned} \tag{5.4.10}$$

On the other hand, the fifth equation in system (5.4.7) can be rewritten as

$$\begin{aligned}
\lambda\zeta_2 =& \alpha\Delta_s^2\zeta_1 - \delta(\Delta_s)^{\frac{1}{2}}\zeta_2 + A_2^*P\mathbf{\Phi} - A_2^*\nabla N_s\zeta_2 - \lambda\gamma_s^+N_s\zeta_2 - \gamma_s^+N_\Phi P\mathbf{\Phi} \\
& - \gamma_s^{+,-}(N_\Phi P\mathbf{\Phi} + \lambda N_s\zeta_2 - N_\Phi\nabla N_s\zeta + N_p\mathbf{G}_f) \\
=& A_2^*P\mathbf{\Phi} - \gamma_s^{+,-}N_\Phi P\mathbf{\Phi} + \alpha\Delta_s^2\zeta_1 - \delta(\Delta_s)^{\frac{1}{2}}\zeta_2 - A_2^*\nabla N_s\zeta_2 \\
& - \lambda\gamma_s^{+,-}N_s\zeta_2 + \gamma_s^{+,-}N_\Phi\nabla N_s\zeta_2 - \gamma_s^{+,-}N_p\mathbf{G}_f + G_s^2
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\lambda\left[I + \gamma_s^{+,-}N_s\right]\zeta_2 =& A_2^*P\mathbf{\Phi} - \gamma_s^{+,-}N_\Phi P\mathbf{\Phi} + \alpha\Delta_s^2\zeta_1 - \delta(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^*\nabla N_s\zeta_2 \\
& + \gamma_s^{+,-}N_\Phi\nabla N_s\zeta_2 - \gamma_s^{+,-}N_p\mathbf{G}_f + G_s^2.
\end{aligned} \tag{5.4.11}$$

Then, the result follows from the identities (5.4.9), (5.4.10) and (5.4.11). □

Theorem 5.4.4. *Let $\alpha \in (0, \alpha^*)$. Then, the adjoint of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $\mathbf{Z}$ is defined by*

$$\mathcal{D}(\mathcal{A}^*) = \left\{M^*(P\mathbf{\Phi}, \zeta_1, \zeta_2) \mid (P\mathbf{\Phi}, \zeta_1, \zeta_2) \in \mathcal{D}(\mathcal{A}^\#)\right\} \text{ and } \mathcal{A}^* = \mathcal{A}^\# M^{-*}.$$

Proof. Let $\mathbf{F}_f \in \mathbf{L}^2(\Omega)$, $F_s^1 \in H^2_{\{0\}}(0, \ell_s)$ and $F_s^2 \in L^2(0, \ell_s)$. Let us consider the system

$$\begin{cases}
\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v}, q) + (\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 = \mathbf{F}_f \ \text{ in } \Omega, \\
\operatorname{div}\mathbf{v} = A_3\eta_1 \ \text{ in } \Omega, \\
\mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_d\setminus\Gamma_s, \quad \sigma(\mathbf{v}, q)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \\
\lambda\eta_1 - \eta_2 = F_s^1 \ \text{ on } (0, \ell_s), \\
\lambda\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 - A_4\eta_1 = -\gamma_s^+q + \gamma_s^-q + F_s^2 \ \text{ on } (0, \ell_s), \\
\eta_1(0) = \partial_x\eta(0) = 0 \ \text{ and } \ \partial_x^2\eta_1(\ell_s) = \partial_x^3\eta_1(\ell_s) = 0.
\end{cases} \tag{5.4.12}$$

For a given $\mathbf{G}_f \in \mathbf{L}^2(\Omega)$, $G_s^1 \in H^2_{\{0\}}(0,\ell_s)$ and $G_s^2 \in L^2(0,\ell_s)$, let us consider the system

$$\begin{cases} \lambda\mathbf{\Phi} - \operatorname{div}\sigma(\mathbf{\Phi},\psi) - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi} + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi} = \mathbf{G}_f \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathbf{\Phi} = \zeta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d\setminus\Gamma_s, \quad \sigma(\mathbf{\Phi},\psi)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi} = 0 \ \text{ on } \Gamma_n, \\ \lambda\zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm}A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*\psi = G_s^1 \ \text{ in } (0,\ell_s), \\ \lambda\zeta_2 + \alpha\Delta_s^2\zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^*\mathbf{\Phi} = -\gamma_s^+\psi + \gamma_s^-\psi + G_s^2 \ \text{ in } (0,\ell_s), \\ \zeta_1(0) = \partial_x\zeta_1(0) = 0 \ \text{ and } \ \partial_x^2\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_2(\ell_s), \ \ \partial_x^3\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s). \end{cases} \tag{5.4.13}$$

After integrating by parts, we obtain the identity (see Appendix C at the end of this chapter)

$$\begin{aligned} &\Big\langle \mathbf{F}_f, \mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)} + \alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1 + \int_0^{\ell_s}F_s^2\zeta_2 \\ =&\Big\langle \mathbf{G}_f, \mathbf{v}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)} + \alpha\int_0^{\ell_s}\Delta G_s^1\cdot\Delta\eta_1 + \int_0^{\ell_s}G_s^2\eta_2. \end{aligned} \tag{5.4.14}$$

where $(\mathbf{v},\eta_1,\eta_2)$ and $(\mathbf{\Phi},\zeta_1,\zeta_2)$ represent the solutions to systems (5.4.12) and (5.4.13), respectively. Then, the identity (5.4.14) can be rewritten as follows:

$$\Big\langle (\mathbf{F}_f, F_s^1, F_s^2)^\top, (\mathbf{\Phi},\zeta_1,\zeta_2)^\top\Big\rangle_{\mathbf{Z}',\mathbf{Z}} = \Big\langle (\mathbf{G}_f, G_s^1, G_s^2)^\top, (\mathbf{v},\eta_1,\eta_2)^\top\Big\rangle_{\mathbf{Z}',\mathbf{Z}} \tag{5.4.15}$$

Let us notice that from Theorem 5.3.5 we have that the triple $(P\mathbf{v},\eta_1,\eta_2)$ satisfies

$$M(\lambda I - \mathcal{A})(P\mathbf{v},\eta_1,\eta_2)^\top = (P\mathbf{F}_f, F_s^1, F_s^2)^\top = (\mathbf{F}_f, F_s^1, F_s^2)^\top, \tag{5.4.16}$$

while on the other hand, according Theorem 5.4.3, we have that the triple $(P\mathbf{\Phi},\zeta_1,\zeta_2)$ satisfies

$$(\lambda M^* - \mathcal{A}^{\#})(P\mathbf{\Phi},\zeta_1,\zeta_2)^\top = (P\mathbf{G}_f, G_s^1, G_s^2)^\top = (\mathbf{G}_f, G_s^1, G_s^2)^\top. \tag{5.4.17}$$

Then, using (5.4.16) and (5.4.17) in the identity (5.4.15), we get

$$\begin{aligned} &\Big\langle M(\lambda I - \mathcal{A})(P\mathbf{v},\eta_1,\eta_2)^\top, (P\mathbf{\Phi},\zeta_1,\zeta_2)^\top\Big\rangle_{\mathbf{Z}',\mathbf{Z}} \\ &\qquad = \Big\langle (\lambda M^* - \widehat{\mathcal{A}}^{\#})(P\mathbf{\Phi},\zeta_1,\zeta_2)^\top, (P\mathbf{v},\eta_1,\eta_2)^\top\Big\rangle_{\mathbf{Z}',\mathbf{Z}}, \end{aligned} \tag{5.4.18}$$

from where we deduce that $\mathcal{A}^* = \mathcal{A}^{\#}M^{-*}$. $\square$

The following result is a consequence of Theorem 5.4.2.

Theorem 5.4.5. *Let $\alpha\in(0,\alpha^*)$ and $\delta\in(\delta^*,1)$. Let us assume that $\lambda\in\mathbb{C}$ and $(\mathbf{G}_f, G_s^1, G_s^2) \in \mathbf{H}^{-\frac12+\alpha,0}_{\Gamma_d}(\Omega)\times H_s$. Then, a quadruple $(\mathbf{\Phi},\psi,\zeta_1,\zeta_2)\in \mathbf{H}^{\frac32+\alpha,2}_{\delta}(\Omega)\times H^{\frac12+\alpha,1}_{\delta}(\Omega)\times H^4_{\{0,\ell_s\}}(0,\ell_s)\times H^2_{\{0\}}(0,\ell_s)$ is a solution of system (5.4.7), if and only if,*

$$\begin{cases} \lambda M^*\begin{pmatrix} P\mathbf{\Phi} \\ \zeta_1 \\ \zeta_2\end{pmatrix} = \mathcal{A}^*M^*\begin{pmatrix} P\mathbf{\Phi} \\ \zeta_1 \\ \zeta_2\end{pmatrix} + \begin{pmatrix} P\mathbf{G}_f \\ G_s^1 - \frac{1}{\alpha}(\Delta_s^2)^{-1}A_3^*N_p\mathbf{G}_f \\ G_s^2 - \gamma_s^{+,-}N_p\mathbf{G}_f\end{pmatrix}, \\ (I-P)\mathbf{\Phi} = -\nabla N_s\zeta_2, \\ \psi = \lambda N_s\zeta_2 + N_\Phi\left(P\mathbf{\Phi} - \nabla N_s\zeta_2\right) + N_p\mathbf{G}_f. \end{cases} \tag{5.4.19}$$

5.5 Resolvent of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and analyticity of the underlying semigroup

We first decompose the unbounded operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ as $\mathcal{A} = \mathcal{A}_1 + B_1 + B_2 + B_3 + B_4$, where

$$\mathcal{A}_1 = \begin{pmatrix} A & (\lambda_f I - A)PL(0, A_3 \cdot) & (\lambda_f I - A)PL(\cdot, 0) \\ 0 & 0 & I \\ 0 & -\alpha\Delta_s^2 & -\gamma B \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}),$$

$$B_1 \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(K_s^{-1} - I)\alpha\Delta_s^2\eta_1 \end{pmatrix}, \quad \mathcal{D}(B_1) = \mathcal{D}(\mathcal{A}),$$

$$B_2 \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(K_s^{-1} - I)\gamma B\eta_2 \end{pmatrix}, \quad \mathcal{D}(B_2) = \mathcal{D}(\mathcal{A}),$$

$$B_3 \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} PA_1\eta_1 + PA_2\eta_2 \\ 0 \\ K_s^{-1}\left(A_4\eta_1 - \gamma_s^{+,-}N_pA_1\eta_1 - \gamma_s^{+,-}N_pA_2\eta_2 + \gamma_s^{+,-}N_{\mathrm{div}}A_3\eta_2\right) \end{pmatrix}, \quad \mathcal{D}(B_3) = \mathcal{D}(\mathcal{A}),$$

$$B_4 \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K_s^{-1}\gamma_s^{+,-}[\Psi N_v\left(P\mathbf{v} + \nabla N_s\eta_2 - \nabla N_{\mathrm{div}}A_3\eta_1\right)] \end{pmatrix}, \quad \mathcal{D}(B_4) = \mathcal{D}(\mathcal{A}).$$

We now state some useful properties of the operators $\mathcal{A}_1$ and B_j $(j = 1, \dots, 4)$.

For $a \in \mathbb{R}$ and $\theta \in (0, \pi)$, we define the sector $\Sigma_{a,\theta}$ by

$$\Sigma_{a,\theta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| < \theta\}.$$

The proof of the following result can be adapted from the proof of Theorem 3.3.8 presented in Chapter 3.

Theorem 5.5.1. *The following assertions hold:*

(i) *There exists $a \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that the sector $\Sigma_{a,\theta}$ is contained in the resolvent set $\rho(\mathcal{A}_1)$ of the unbounded operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$. Moreover, there exists a positive constant C such that*

$$\|(\lambda I - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathbf{Z})} \leq \frac{C}{|\lambda - a|}, \quad \text{for all } \lambda \in \Sigma_{a,\theta} \setminus \{0\}. \tag{5.5.1}$$

(ii) *The domain $\mathcal{D}(\mathcal{A})$ of the fluid-structure operator defined in (5.3.39) is dense in $\mathbf{Z}$.*

(iii) *The unbounded operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$.*

Adapting the proof from [Ray10, Lemma 3.9] and subsequently applying [Paz83, Chapter 2, Corollary 6.11], we obtain the following result:

Proposition 5.5.1. *The operator $(B_1, \mathcal{D}(\mathcal{A}))$ and $(B_2, \mathcal{D}(\mathcal{A}))$ are $\mathcal{A}_1-$bounded with $\mathcal{A}_1-$bound equal to zero.*

Proposition 5.5.2. *The operator $(B_3, \mathcal{D}(\mathcal{A}))$ is $\mathcal{A}_1-$bounded with $\mathcal{A}_1-$bound equal to zero.*

Proof. Let us assume that the assertion is false. Then, there exist $\varepsilon > 0$ and a sequence $(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})_k$ in $\mathcal{D}(\mathcal{A})$ such that

$$\left\|B_3(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} > \varepsilon \left\|\mathcal{A}_1(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} + k\left\|(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}}. \tag{5.5.2}$$

Let us assume without loss of generality that $\left\|B_3(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} = 1$. From (5.5.2) it follows that

$$\left\|(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} \xrightarrow[k\to+\infty]{} 0 \ \text{ in } \mathbf{Z}. \tag{5.5.3}$$

Then, thanks to the continuity of the Leray projector P from $\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)$ into $\mathbf{V}^{-\frac{1}{2}+\alpha,0}_{n,\Gamma_d}(\Omega)$, along with Proposition 5.3.2, Lemma 3.3.2 and the fact that $N_{\text{div}} \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$, $N_p \in \mathcal{L}(\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega), H^{\frac{1}{2}+\alpha,1}(\Omega))$, we have

$$\begin{aligned}
\left\|B_3(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} &\le \|PA_1\eta_1 + PA_2\eta_2\|_{\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)} \\
&\quad + \left\|K_s^{-1}\left(A_4\eta_{1,k} - \gamma_s^{+,-}N_pA_1\eta_{1,k}\right)\right\|_{H^\alpha(0,\ell_s)} \\
&\quad + \left\|K_s^{-1}\left(-\gamma_s^{+,-}N_pA_2\eta_2 + \gamma_s^{+,-}N_{\text{div}}A_3\eta_2\right)\right\|_{H^\alpha(0,\ell_s)} \\
&\le C\,\|(\eta_{1,k}, \eta_{2,k})\|_{H^2_{\{0\}}(0,\ell_s)\times L^2(0,\ell_s)}\,.
\end{aligned}$$

Finally, we notice that the last estimate yields a contradiction with the fact that

$$\left\|B_3(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k})^\top\right\|_{\mathbf{Z}} = 1.$$

This completes the proof. □

The proof of the next proposition can be adapted from the proof of Theorem 3.3.9 of Chapter 3.

Proposition 5.5.3. *The operator $(B_4, \mathcal{D}(\mathcal{A}))$ is compact on $\mathbf{Z}$. Furthermore, the operator B_4 is A_1-bounded with A_1-bound equal to zero.*

We now state the main theorem of this section.

Theorem 5.5.2. *The following assertions hold:*

(i) *There exist $a \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that sector $\Sigma_{a,\theta}$ is contained in the resolvent set $\rho(\mathcal{A})$ of the fluid-structure operator (5.3.39) and the following estimate holds:*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathbf{Z})} \le \frac{C}{|\lambda|}, \quad \textit{for all } \lambda \in \Sigma_{a,\theta}\setminus\{0\}, \tag{5.5.4}$$

for a certain positive constant C. Moreover, the fluid-structure operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$.

(ii) *The resolvent of the fluid-structure operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is compact on $\mathbf{Z}$.*

Proof.

(i) The first part of the assertion is a direct application of [EN06, Lemma 2.6, p.127] in combination with Propositions 5.5.1, 5.5.2 and 5.5.3.

The fact that the fluid-structure operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup follows from the assertion (iii) of Theorem 5.5.1, Propositions 5.5.1, 5.5.2 and 5.5.3 and [Paz83, Chapter 3, Theorem 2.1] (see also [EN06, Chapter III, Theorem 2]).

(ii) The compactness of the resolvent is a consequence of the compact embedding $\mathbf{V}^{\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)\times\mathcal{D}(A_s) \hookrightarrow \mathbf{Z}$.

□

5.6 Eigenvalue problems

5.6.1 Direct eigenvalue problem

Let us consider the direct eigenvalue problem

$$\begin{cases} \lambda \in \mathbb{C}, \quad (\mathbf{v}, q, \eta_1, \eta_2) \in \mathbf{H}^\alpha_\delta, \\ \lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, q) + (\mathbf{u}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 = 0 \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} = A_3\eta_1 \ \text{ in } \Omega, \\ \mathbf{v} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_d \setminus \Gamma_s, \quad \sigma(\mathbf{v}, q)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \\ \lambda \eta_1 - \eta_2 = F^1_s \ \text{ in } (0, \ell_s), \\ \lambda \eta_2 + \alpha \Delta^2 \eta_1 + \gamma (\Delta^2_s)^{\frac{1}{2}} \eta_2 - A_4 \eta_1 + \gamma^+_s q - \gamma^-_s q = 0 \ \text{ in } (0, \ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ on } \{0\} \ \text{ and } \ \partial^2_{x_1}\eta_1 = 0, \partial^3_{x_1}\eta_1 = 0 \ \text{ on } \{\ell_s\}, \end{cases} \tag{5.6.1}$$

and the eigenvalue problem associated to the fluid-structure operator $\mathcal{A}$, namely,

$$\lambda \in \mathbb{C}, \quad (P\mathbf{v}, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}), \quad \lambda (P\mathbf{v}, \eta_1, \eta_2)^\top = \mathcal{A}(P\mathbf{v}, \eta_1, \eta_2)^\top. \tag{5.6.2}$$

The following result is a consequence of Theorem 5.3.5.

Theorem 5.6.1. *A couple $(\lambda, (\mathbf{v}, q, \eta_1, \eta_2)) \in \mathbb{C} \times \mathbf{H}^\alpha_\delta$ is a solution of the eigenvalue problem (5.6.1) if and only if, $(\lambda, (P\mathbf{v}, \eta_1, \eta_2)) \in \mathbb{C} \times \mathcal{D}(\mathcal{A})$ is a solution of (5.6.2) and*

$$\begin{aligned} &(I - P)\mathbf{v} = \nabla N_{\mathrm{div}} A_3 \eta_1 - \nabla N_s \eta_2, \\ &\qquad q = -\lambda N_{\mathrm{div}} A_3 \eta_1 + \lambda N_s \eta_2 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v (P\mathbf{v} - \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1). \end{aligned}$$

Definition 5.6.1. *A triplet $(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k}) \in \mathcal{D}(\mathcal{A})$ is a generalized eigenfunction for problem (5.6.2) of order $k \geq 1$ associated to a solution $(\lambda, (P\mathbf{v}_0, \eta_{1,0}, \eta_{2,0}))$ of (5.6.2) if $(\lambda, (P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k}))$ is obtained by solving the chain of equations*

$$(\lambda I - \mathcal{A})(P\mathbf{v}_j, \eta_{1,j}, \eta_{2,j})^\top = -(P\mathbf{v}_{j-1}, \eta_{1,j-1}, \eta_{2,j-1})^\top \ \textit{ for } 1 \leq j \leq k.$$

Definition 5.6.2. *A quadruplet $(\mathbf{v}_k, q_k, \eta_{1,k}, \eta_{2,k}) \in \mathbf{H}^\alpha_\delta$ is a generalized eigenfunction for problem (5.6.1) of order $k \geq 1$ associated to a solution $(\lambda, (\mathbf{v}_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (5.6.1) if $(\mathbf{v}_k, q_k, \eta_{1,k}, \eta_{2,k})$ is obtained by solving, for $1 \leq j \leq k$, the chain of systems*

$$\begin{cases} \lambda \mathbf{v}_j - \operatorname{div} \sigma(\mathbf{v}_j, q_j) + (\mathbf{u}_s \cdot \nabla)\mathbf{v}_j + (\mathbf{v}_j \cdot \nabla)\mathbf{u}_s - A_1\eta_{1,j} - A_2\eta_{2,j} = -\mathbf{v}_{j-1} \ \textit{ in } \Omega, \\ \operatorname{div} \mathbf{v}_j = A_3\eta_{1,j} \ \textit{ in } \Omega, \\ \mathbf{v}_j = \eta_{2,j} \vec{\mathbf{e}}_2 \ \textit{ on } \Gamma_s, \quad \mathbf{v}_j = 0 \ \textit{ on } \Gamma_d \setminus \Gamma_s, \quad \sigma(\mathbf{v}_j, q_j)\mathbf{n} = 0 \ \textit{ on } \Gamma_n, \\ \lambda \eta_{1,j} - \eta_{2,j} = -\eta_{1,j-1} \ \textit{ in } (0, \ell_s), \\ \lambda \eta_{2,j} + \alpha \Delta^2 \eta_{1,j} + \gamma (\Delta^2_s)^{\frac{1}{2}} \eta_{2,j} - A_4 \eta_{1,j} + \gamma^+_s q_j - \gamma^-_s q_j = -\eta_{2,j-1} \ \textit{ in } (0, \ell_s), \\ \eta_{1,j} = 0, \partial_{x_1}\eta_{1,j} = 0 \ \textit{ on } \{0\} \ \textit{ and } \ \partial^2_{x_1}\eta_{1,j} = 0, \partial^3_{x_1}\eta_{1,j} = 0 \ \textit{ on } \{\ell_s\}. \end{cases} \tag{5.6.3}$$

Theorem 5.6.2. *A quadruplet $(\mathbf{v}_k, q_k, \eta_{1,k}, \eta_{2,k}) \in \mathbf{H}^\alpha_\delta$ is a generalized eigenfunction associated with a solution $(\lambda, (\mathbf{v}_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (5.6.1) if and only if the triplet $(P\mathbf{v}_k, \eta_{1,k}, \eta_{2,k}) \in \mathcal{D}(\mathcal{A})$ is a generalized eigenfuncion for (5.6.2), which is associated with a solution $(\lambda, (P\mathbf{v}_0, \eta_{1,0}, \eta_{2,0})$ and*

$$\begin{aligned} &(I - P)\mathbf{v}_k = \nabla N_{\mathrm{div}} A_3 \eta_{1,k} - \nabla N_s \eta_{2,k}, \\ &\qquad q_k = -\lambda N_{\mathrm{div}} A_3 \eta_{1,k} + \lambda N_s \eta_{2,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} \\ &\qquad\qquad + N_v (P\mathbf{v}_k - \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k}) + N_s \eta_{2,k-1} - N_{\mathrm{div}} A_3 \eta_{1,k-1}. \end{aligned}$$

Proof. According to Theorem 5.3.5, we first observe that $(\mathbf{v}_k, q_k, \eta_{1,k}, \eta_{2,k}) \in \mathbf{H}^\alpha_\delta$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (\mathbf{v}_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (5.6.1) if and only if

$$
\begin{aligned}
(\lambda I - \mathcal{A}) \begin{pmatrix} P\mathbf{v}_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} &= -M^{-1} \begin{pmatrix} P\mathbf{v}_{k-1} \\ \eta_{1,k-1} \\ -\gamma_s^{+,-} N_p \mathbf{v}_{k-1} \end{pmatrix}, \\
(I-P)\mathbf{v}_k &= \nabla N_{\mathrm{div}} A_3 \eta_{1,k} - \nabla N_s \eta_{2,k}, \\
q_k &= -\lambda N_{\mathrm{div}} A_3 \eta_{1,k} + \lambda N_s \eta_{2,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} \\
&\quad + N_v (P\mathbf{v}_k - \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k}) - N_p \mathbf{v}_{k-1},
\end{aligned}
$$

where $(\mathbf{v}_{k-1}, q_{k-1}, \eta_{1,k-1}, \eta_{2,k-1})$ is a generalized eigenfunction of order $k-1$. Then, using the identity $N_p \mathbf{v}_{k-1} = N_{\mathrm{div}} A_3 \eta_{1,k-1} - N_s \eta_{2,k-1}$ and the explicit expression of M^{-1} given in (5.3.38), we get

$$
\begin{aligned}
(\lambda I - \mathcal{A}) \begin{pmatrix} P\mathbf{v}_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} &= - \begin{pmatrix} P\mathbf{v}_{k-1} \\ \eta_{1,k-1} \\ \eta_{2,k-1}, \end{pmatrix} \\
(I-P)\mathbf{v}_k &= \nabla N_{\mathrm{div}} A_3 \eta_{1,k} - \nabla N_s \eta_{2,k}, \\
q_k &= -\lambda N_{\mathrm{div}} A_3 \eta_{1,k} + \lambda N_s \eta_{2,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} \\
&\quad + N_v (P\mathbf{v}_k - \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k}) \\
&\quad + N_s \eta_{2,k-1} - N_{\mathrm{div}} A_3 \eta_{1,k-1}.
\end{aligned}
$$

This completes the proof. □

5.6.2 Adjoint eigenvalue problem

We consider the adjoint eigenvalue problem

$$
\begin{cases}
\lambda \in \mathbb{C}, \quad (\mathbf{\Phi}, \Psi, \zeta_1, \zeta_2) \in \mathbf{H}^\alpha_\delta, \\
\lambda \mathbf{\Phi} - \operatorname{div} \sigma(\mathbf{\Phi}, \psi) - (\mathbf{u}_s \cdot \nabla)\mathbf{\Phi} + (\nabla \mathbf{u}_s)^\top \mathbf{\Phi} = 0 \ \text{ in } \Omega, \\
\operatorname{div} \mathbf{\Phi} = 0 \ \text{ in } \Omega, \\
\mathbf{\Phi} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d \setminus \Gamma_s, \quad \sigma(\mathbf{\Phi}, \psi)\mathbf{n} + \mathbf{u}_s \cdot \mathbf{n} \mathbf{\Phi} = 0 \ \text{ on } \Gamma_n, \\
\lambda \zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm} A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1} A_1^* \mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1} A_3^* \psi = G_s^1 \ \text{ in } (0, \ell_s), \\
\lambda \zeta_2 + \alpha \Delta_s^2 \zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}} \zeta_2 - A_2^* \mathbf{\Phi} = -\gamma_s^+ \psi + \gamma_s^- \psi + G_s^2 \ \text{ in } (0, \ell_s), \\
\zeta_1(0) = \partial_x \zeta_1(0) = 0 \ \text{ and } \ \partial_x^2 \zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha} \zeta_2(\ell_s), \ \ \partial_x^3 \zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha} \zeta_{2,x}(\ell_s),
\end{cases}
\tag{5.6.4}
$$

and the adjoint eigenvalue problem associated to the adjoint of the fluid-strucure operator

$$
\lambda \in \mathbb{C}, \ \ (P\mathbf{\Phi}, \zeta_1, \zeta_2) \in D(\mathcal{A}^*), \ \ \lambda (P\mathbf{\Phi}, \zeta_1, \zeta_2)^\top = \mathcal{A}^* (P\mathbf{\Phi}, \zeta_1, \zeta_2)^\top. \tag{5.6.5}
$$

The following result is a consequence of Theorem 5.4.5.

Theorem 5.6.3. *A couple $(\lambda, (\mathbf{\Phi}, \psi, \zeta_1, \zeta_2)) \in \mathbb{C} \times \mathbf{H}^\alpha_\delta$ is a solution of the eigenvalue problem (5.6.4) if and only if $(\lambda, M^*(P\mathbf{\Phi}, \psi, \zeta_1, \zeta_2)) \in \mathbb{C} \times D(\mathcal{A}^*)$ is a solution of (5.6.5) and*

$$
\begin{aligned}
(I-P)\mathbf{\Phi} &= -\nabla N_s \zeta_2, \\
\psi &= \lambda N_s \zeta_2 + N_\Phi \left(P\mathbf{\Phi} - \nabla N_s \zeta_2\right).
\end{aligned}
$$

Definition 5.6.3. *A triplet $(P\mathbf{\Phi}_k, \zeta_{1,k}, \zeta_{2,k}) \in \mathcal{D}(\mathcal{A})$ is a generalized eigenfunction for problem (5.6.5) of order $k \geq 1$ associated to a solution $(\lambda, (P\mathbf{\Phi}_0, \zeta_{1,0}, \zeta_{2,0}))$ of (5.6.5) if $(\lambda, (P\mathbf{\Phi}_k, \zeta_{1,k}, \zeta_{2,k}))$ is obtained by solving the chain of equations*

$$
(\lambda I - \mathcal{A}^*)(P\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})^\top = -(P\mathbf{\Phi}_{j-1}, \zeta_{1,j-1}, \zeta_{2,j-1})^\top \ \ \textit{for } 1 \leq j \leq k.
$$

Definition 5.6.4. *A quadruplet* $(\mathbf{\Phi}_j, \psi_j, \zeta_{1,j}, \zeta_{2,j}) \in \mathbf{H}^\alpha_\delta$ *is a generalized eigenfunction for problem* (5.6.4) *of order* $k \geq 1$ *associated to a solution* $(\lambda, (\mathbf{\Phi}_0, \psi_0, \zeta_{1,0}, \zeta_{2,0}))$ *of* (5.6.4) *if* $(\mathbf{\Phi}_k, \psi_k, \zeta_{1,k}, \zeta_{2,k})$ *is obtained by solving, for* $1 \leq j \leq k$, *the chain of systems*

$$\begin{cases} \lambda\mathbf{\Phi}_j - \operatorname{div}\sigma(\mathbf{\Phi}_j, \psi_j) - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi}_j + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi}_j = -\mathbf{\Phi}_{j-1} \ \ \textit{in } \Omega, \\ \operatorname{div}\mathbf{\Phi}_j = 0 \ \ \textit{in } \Omega, \\ \mathbf{\Phi}_j = \zeta_{2,j}\vec{\mathbf{e}}_2 \ \ \textit{on } \Gamma_s, \ \ \mathbf{\Phi}_j = 0 \ \ \textit{on } \Gamma_d\setminus\Gamma_s, \ \ \sigma(\mathbf{\Phi}_j,\psi_j)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi}_j = 0 \ \ \textit{on } \Gamma_n, \\ \lambda\zeta_{1,j} + \zeta_{2,j} - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm}A_3)^*)\zeta_{2,j} - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*\psi_j = -\zeta_{j-1} \ \textit{in } (0,\ell_s), \\ \lambda\zeta_{2,j} - \alpha\Delta_s^2\zeta_{1,j} + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_{2,j} - A_2^*\mathbf{\Phi}_j + \gamma_s^+\psi_j - \gamma_s^-\psi_j = -\zeta_{2,j-1} \ \ \textit{in } (0,\ell_s), \\ \zeta_{1,j}(0) = \partial_x\zeta_{1,j}(\ell_s) = 0, \ \ \partial_x^2\zeta_{1,j}(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_{2,j}(\ell_s) \ \textit{and} \ \partial_x^3\zeta_{1,j}(\ell_s) = \dfrac{2\nu e}{\alpha}\partial_x\zeta_{2,j}(\ell_s). \end{cases} \tag{5.6.6}$$

Theorem 5.6.4. *A quadruplet* $(\mathbf{\Phi}_k, \psi_k, \zeta_{1,k}, \zeta_{2,k}) \in \mathbf{H}^\alpha_\delta$ *is a generalized eigenfunction of order* $k \geq 1$ *associated with a solution* $(\lambda, (\mathbf{\Phi}_0, \psi_0, \zeta_{1,0}, \zeta_{2,0}))$ *of* (5.6.4), *if and only if,* $M^*(P\mathbf{\Phi}_k, \psi_k, \zeta_{1,k}, \zeta_{2,k}) \in D(\mathcal{A}^*)$ *is a generalized eigenfunction of order* $k \geq 1$ *associated with a solution* $(\lambda, (P\mathbf{\Phi}_0, \zeta_{1,0}, \zeta_{2,0}))$ *of* (5.6.5) *and*

$$(I-P)\mathbf{\Phi}_k = -\nabla N_s\zeta_{2,k},$$
$$\psi_k = \lambda N_s\zeta_{2,k} + N_\Phi\left(P\mathbf{\Phi}_k - \nabla N_s\zeta_{2,k}\right) + N_s\zeta_{2,k-1}.$$

Proof. According to Theorem 5.4.5, we have that $(\mathbf{\Phi}_k, \psi_k, \zeta_{1,k}, \zeta_{2,k}) \in \mathbf{H}^\alpha_\delta$ is a generalized eigenfunction of order $k \geq 1$ with a solution $(\lambda, (\mathbf{\Phi}_0, \psi_0, \zeta_{1,0}, \zeta_{2,0}))$ of (5.6.4), if and only if,

$$\begin{aligned} (\lambda I - \mathcal{A}^*)M^*\begin{pmatrix} P\mathbf{\Phi}_k \\ \zeta_{1,k} \\ \zeta_{2,k}\end{pmatrix} &= \begin{pmatrix} -P\mathbf{\Phi}_{k-1} \\ -\zeta_{1,k-1} - \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p(-\mathbf{\Phi}_{k-1}) \\ -\mathbf{\Phi}_{k-1} + \gamma_s^{+,-}N_p(-\mathbf{\Phi}_{k-1})\end{pmatrix}, \\ (I-P)\mathbf{\Phi}_k &= -\nabla N_s\zeta_{2,k}, \\ \psi_k &= \lambda N_s\zeta_{2,k} + N_\Phi\left(P\mathbf{\Phi}_k - \nabla N_s\zeta_{2,k}\right) - N_p\mathbf{\Phi}_k, \end{aligned} \tag{5.6.7}$$

where $(\mathbf{\Phi}_{k-1}, \psi_{k-1}, \zeta_{1,k-1}, \zeta_{2,k-1}) \in \mathbf{H}^\alpha_\delta$ is a generalized eigenfunction of order $k-1$. Then, using the identity $N_p(-\mathbf{\Phi}_k) = N_s\zeta_{2,k}$ and the explicit expression of M^* (see Proposition 5.4.1), the right-hand side of the first equation in (5.6.7) can be rewritten as

$$\begin{aligned} \begin{pmatrix} -P\mathbf{\Phi}_{k-1} \\ -\zeta_{1,k-1} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p\mathbf{\Phi}_{k-1} \\ -\mathbf{\Phi}_{k-1} + \gamma_s^{+,-}N_p\mathbf{\Phi}_{k-1}\end{pmatrix} &= M^*M^{-*}\begin{pmatrix} -P\mathbf{\Phi}_{k-1} \\ -\zeta_{1,k-1} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*N_p\mathbf{\Phi}_{k-1} \\ -\zeta_{2,k-1} + \gamma_s^{+,-}N_p\mathbf{\Phi}_{k-1}\end{pmatrix} \\ &= -M^*\begin{pmatrix} P\mathbf{\Phi}_{k-1} \\ \zeta_{1,k-1} \\ \zeta_{2,k-1}\end{pmatrix}. \end{aligned}$$

Finally, using the last equality together with the fact that $N_p(-\mathbf{\Phi}_k) = N_s\zeta_{2,k}$ in (5.6.7), we conclude the result. $\square$

5.7 Stabilization of the linearized system

Thanks to assertion (ii) of Theorem 5.5.2 we know that the spectrum of $\mathcal{A}$ is only pointwise spectrum. Let $(\lambda_j)_{j\in\mathbb{N}^*}$ be the eigenvalues of $\mathcal{A}$. For a given eigenvalue λ_j of $\mathcal{A}$,

$$G_{\mathbb{R}}(\lambda_j) = \operatorname{span}\{\Re G_{\mathbb{C}}(\lambda_j) \cup \Im G_{\mathbb{C}}(\lambda_j)\} \text{ and } G^*_{\mathbb{R}}(\lambda_j) = \operatorname{span}\{\Re G^*_{\mathbb{C}}(\lambda_j) \cup \Im G^*_{\mathbb{C}}(\lambda_j)\}$$

denote the real generalized eigenspaces of $\mathcal{A}$ and $\mathcal{A}^*$, respectively. Here, $G_{\mathbb{C}}(\lambda_j)$ and $G^*_{\mathbb{C}}(\lambda_j)$ denote the complex generalized eigenspaces of $\mathcal{A}$ and $\mathcal{A}^*$, respectively. Let ω be a real number such that $-\omega \notin \{\Re\lambda_j \mid j \in \mathbb{N}^*\}$. We define the unstable spaces

$$\mathbf{Z}_u = \bigoplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \quad \mathbf{Z}^*_u = \bigoplus_{j\in J_u} G^*_{\mathbb{R}}(\lambda_j),$$

where J_u is a finite subset of $\mathbb{N}^*$ given by $J_u := \{j \in \mathbb{N}^* \mid \Re\lambda_j \geq -\omega\}$. Let us notice that $\mathbf{Z}_u \subset \mathcal{D}(\mathcal{A})$. There exist two subspaces $\mathbf{Z}_s$ and $\mathbf{Z}^*_s$, invariant under $(e^{t\mathcal{A}})_{t\geq 0}$ and $(e^{t\mathcal{A}^*})_{t\geq 0}$, respectively, such that

$$\mathbf{Z} = \mathbf{Z}_u \oplus \mathbf{Z}_s \quad \text{and} \quad \mathbf{Z}^* = \mathbf{Z}^*_u \oplus \mathbf{Z}^*_s.$$

We denote by Π_u the projection from $\mathbf{Z}$ onto $\mathbf{Z}_u$ along $\mathbf{Z}_s$ and by Π_s the projection from $\mathbf{Z}$ onto $\mathbf{Z}_s$ along $\mathbf{Z}_u$. We also denote by N_u the dimension of the subspace $\mathbf{Z}_u$.

We recall that $\mathbf{Z}_0 = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \times H_s$ and $\mathbf{H}_0 = \mathbf{L}^2(\Omega) \times H_s$, which are endowed with the inner product

$$\left\langle (\mathbf{u}, \eta_1, \eta_2)^\top, (\mathbf{v}, \zeta_1, \zeta_2)^\top \right\rangle_{\mathbf{H}_0} = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)} + \int_0^{\ell_s} (\alpha\eta_{1,xx}\zeta_{1,xx} + \eta_2\zeta_2)\, dx. \tag{5.7.1}$$

The following proposition plays an important role in our analysis, as it allows us to establish the link between the operator equation and the PDE formulation (see Proposition 5.7.4). The proof is presented in the Appendix D at the end of this chapter.

Proposition 5.7.1. *For all* $\tilde{\mathbf{v}} = P\mathbf{v} + \nabla N_{\text{div}} A_3\eta_1 - \nabla N_s\eta_2$ *and* $\widetilde{\mathbf{\Phi}} = P\mathbf{\Phi} - \nabla N_s\zeta_2$, *with* $(P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{Z}_0$ *and* $(P\mathbf{\Phi}, \zeta_1, \zeta_2) \in \mathbf{Z}^*_0$, *we have*

$$\left\langle (\tilde{\mathbf{v}}, \eta_1, \eta_2)^\top, (\widetilde{\mathbf{\Phi}}, \zeta_1, \zeta_2)^\top \right\rangle_{\mathbf{Z}_0} = \left\langle (P\mathbf{v}, \eta_1, \eta_2)^\top, M^*(P\mathbf{\Phi}, \zeta_1, \zeta_2)^\top \right\rangle_{\mathbf{Z}_0}. \tag{5.7.2}$$

The following result establishes the existence of biorthogonal bases of $\mathbf{Z}_u$ and $\mathbf{Z}^*_u$, which allow us to obtain an expression for the projection Π_u.

Proposition 5.7.2. *There exists two families* $\{(\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})\}_{1\leq i\leq N_u}$ *and* $\{(\mathbf{\Phi}_i, \zeta_{1,i}, \zeta_{2,i})\}_{1\leq i\leq N_u}$ *belonging to* $\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega) \times H^4_{\{0,\ell_s\}}(0,\ell_s) \times H^2_{\{0\}}(0,\ell_s)$ *satisfying the following assertions:*

(a) *The families* $\{(P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})\}_{1\leq i\leq N_u}$ *and* $\left\{M^*(P\mathbf{\Phi}_i, \zeta_{1,i}, \zeta_{2,i})^\top\right\}_{1\leq i\leq N_u}$ *are basis of* $\mathbf{Z}_u$ *and* $\mathbf{Z}^*_u$, *respectively. Moreover,*

$$\left\langle (\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top, (\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})^\top \right\rangle_{\mathbf{Z}_0} = \delta_{i,j},$$
$$\left\langle (P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top, M^*(P\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})^\top \right\rangle_{\mathbf{Z}_0} = \delta_{i,j}.$$

(b) *For all* $(\mathbf{v}, \eta_1, \eta_2)^\top \in \mathbf{Z}_0$, *we have that*

$$\Pi_u(\mathbf{v}, \eta_1, \eta_2)^\top = \sum_{i=1}^{N_u} \left\langle (\mathbf{v}, \eta_1, \eta_2)^\top, M^*(P\mathbf{\Phi}_i, \zeta_{1,i}, \zeta_{2,i})^\top \right\rangle_{\mathbf{Z}_0} (P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top.$$

Furthermore, if $(\mathbf{v}, \eta_1, \eta_2)^\top \in \mathbf{Z}$,

$$\Pi_u(\mathbf{v}, \eta_1, \eta_2)^\top = \sum_{i=1}^{N_u} \left\langle (\mathbf{v}, \eta_1, \eta_2)^\top, M^*(P\mathbf{\Phi}_i, \zeta_{1,i}, \zeta_{2,i})^\top \right\rangle_{\mathbf{Z},\mathbf{Z}'} (P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top.$$

(c) *For all* $1 \leq i \leq N_u$, *there exists* $q_i \in H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)$ *such that* $(\mathbf{v}_i, q_i, \eta_{1,i}, \eta_{2,i}) \in \mathbf{H}^\alpha_\delta$ *is a real* *or imaginary part of a generalized eigenfunction of* (5.6.1)*. Similarly, there exists* $\psi_i \in H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ *such that* $(\mathbf{\Phi}_i, \psi_i, \zeta_{1,i}, \zeta_{2,i}) \in \mathbf{H}_\delta^\alpha$ *is a real or imaginary part of a generalized eigenfunction of* (5.6.4)*.*

Proof. Let us first notice that, since $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup (see assertion (i) of Theorem 5.5.2) with compact resolvent (see assertion (ii) of Theorem 5.5.2), the following assertions hold (see [Fur01, Lemma 6.2]):

- There exists a basis $\{(\tilde{\mathbf{v}}_i, \eta_{1,i}, \eta_{2,i})\}_{1\le i\le N_u} \subset \mathbf{Z}_0$ of $\mathbf{Z}_u$ consisting of the real or imaginary parts of eigenfunctions or generalized eigenfunctions of $\mathcal{A}$.
- There exists a basis $\left\{M^*(\widetilde{\mathbf{\Phi}}_i, \zeta_{1,i}, \zeta_{2,i})^\top\right\}_{1\le i\le N_u} \subset \mathbf{Z}_0^*$ of $\mathbf{Z}_u^*$ consisting of the real or imaginary parts of eigenfunctions or generalized eigenfunctions of $\mathcal{A}^*$.
- The families $\{(\tilde{\mathbf{v}}_i, \eta_{1,i}, \eta_{2,i})\}_{1\le i\le N_u}$ and $\left\{M^*(\widetilde{\mathbf{\Phi}}_i, \zeta_{1,i}, \zeta_{2,i})^\top\right\}_{1\le i\le N_u}$ satisfy:

$$\left\langle (\tilde{\mathbf{v}}_i, \eta_{1,i}, \eta_{2,i})^\top, M^*(\widetilde{\mathbf{\Phi}}_j, \zeta_{1,j}, \zeta_{2,j})^\top \right\rangle_{\mathbf{Z}_0} = \delta_{i,j},$$

$$\Pi_u(\mathbf{v}, \eta_1, \eta_2)^\top = \sum_{i=1}^{N_u} \left\langle (\mathbf{v}, \eta_1, \eta_2)^\top, M^*(\widetilde{\mathbf{\Phi}}_i, \zeta_{1,i}, \zeta_{2,i}) \right\rangle_{\mathbf{Z}_0} (\tilde{\mathbf{v}}_i, \eta_{1,i}, \eta_{2,i})^\top.$$

Then, by setting $\mathbf{v}_i = \widehat{\mathbf{v}}_i + \nabla N_{\mathrm{div}} A_2 \eta_{1,i} - \nabla N_s \eta_{2,i}$ and $\mathbf{\Phi}_i = \widehat{\mathbf{\Phi}}_i - \nabla N_s \zeta_{2,i}$, we obtain that $P\mathbf{v}_i = \widehat{\mathbf{v}}_i$ and $P\mathbf{\Phi}_i = \widehat{\mathbf{\Phi}}_i$. This fact, together with Proposition 5.7.1, allows us to deduce assertion (a) and the first part of assertion (b). The second part of assertion (b) follows from the density of $\mathbf{Z}_0$ in $\mathbf{Z}$. The assertion (c) is a consequence of Theorems 5.6.2 and 5.6.4. ☐

For a given $j \in J_u$, we set

$$E(\lambda_j) := \{(\mathbf{\Phi}, \psi, \zeta_1, \zeta_2) \in \mathbf{H}_\delta^\alpha \mid (\lambda_j, (\mathbf{\Phi}, \psi, \zeta_1, \zeta_2)) \text{ is solution to the problem (5.6.4)}\}.$$

We denote by $\left\{(\mathbf{\Phi}_j^k, \psi_j^k, \zeta_{1,j}^k, \zeta_{2,j}^k)\right\}_{1\le k\le N_j}$ a basis of $E(\lambda_j)$. Here, $N_j = \dim E(\lambda_j)$. We assume that $(w_i)_{1\le i\le N_c} \subset H^2_{\{0\}}(0, \ell_s)$ is such that

$$\operatorname{span}\{w_i \mid 1 \le i \le N_c\} = \operatorname{span}\left\{\Re\zeta_{2,j}^k, \Im\zeta_{2,j}^k \mid j \in J_u, 1 \le k \le N_j\right\}. \tag{5.7.3}$$

Let us now consider the system

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div}\sigma(\mathbf{v}, q) + (\mathbf{u}_s \cdot \nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega\mathbf{v} = 0 \ \text{ in } Q^\infty, \\ \operatorname{div}\mathbf{v} = A_3\eta_1 \ \text{ in } Q^\infty, \\ \mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma_d^\infty \setminus \Sigma_s^\infty, \ \ \sigma(\mathbf{v}, q)\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty, \\ \partial_t \eta_1 - \eta_2 - \omega\eta_1 = 0 \ \text{ in } (0,\infty)\times(0,\ell_s), \\ \partial_t\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 - A_4\eta_1 - \omega\eta_2 = -\gamma_s^+ q + \gamma_s^- q \\ \qquad\qquad + \sum_{i=1}^{N_c} f_i w_i \ \text{ in } (0,\infty)\times(0,\ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ on } (0,\infty)\times\{0\} \ \text{ and } \ \partial_{x_1}^2\eta_1 = 0, \partial_{x_1}^3\eta_1 = 0 \ \text{ on } (0,\infty)\times\{\ell_s\}. \end{cases} \tag{5.7.4}$$

From Theorem 5.3.5 we deduce that system (5.7.4) can be rewritten as follows:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A} + \omega I)\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\mathbf{f}, \ \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} P\mathbf{v}^0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \\ (I-P)\mathbf{v} = \nabla N_{\mathrm{div}} A_3\eta_1 - \nabla N_s\eta_2, \end{cases} \tag{5.7.5}$$

where $\mathcal{B}\mathbf{f} = \sum_{i=1}^{N_c} f_i(0, 0, (I + \gamma_s^{+,-} N_s)^{-1} w_i)^\top$. We now introduce the matrix $\mathcal{B}_u$:

$$\mathcal{B}_u = \left[\mathcal{B}_u^{i,j}\right]_{\substack{1\le i\le N_u,\\ 1\le j\le N_c}}, \quad \mathcal{B}_u^{i,j} = \int_0^{\ell_s} w_i \zeta_{2,j}, \tag{5.7.6}$$

where $\zeta_{2,j}$ is such that $(\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})_{1\le j\le N_u}$ is the family defined in Proposition 5.7.2. Let us notice that for $\mathcal{A}_{s,\omega} = \Pi_s(\mathcal{A} + \omega I)$, we have

$$\|e^{t\mathcal{A}_{s,\omega}}\|_{\mathcal{L}(\mathbf{Z})} \le Ce^{-\varepsilon_s t} \ \ \forall t > 0, \ \ 0 < \varepsilon_s < \mathrm{dist}(\Re\,\mathrm{spec}(\mathcal{A}_{s,\omega}), 0).$$

The following proposition, whose proof is presented in Appendix E, allows us to reduce the analysis of the stabilization of the linearized system (5.2.13) to the study of the stabilization of the projected system onto $\mathbf{Z}_u$.

Proposition 5.7.3. *The triplet $(P\mathbf{v}, \eta_1, \eta_2)^\top \in \mathcal{D}(\mathcal{A})$ is the solution of the first equation of (5.7.5), if and only if,*

$$\zeta_u = \left(\left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0}\right)_{1\le j\le N_u} \quad \text{and} \quad (\mathbf{v}_s, \eta_{1,s}, \eta_{2,s}) := \Pi_s(P\mathbf{v}, \eta_1, \eta_2)^\top,$$

satisfy

$$\begin{cases} \dfrac{d}{dt}\zeta_u = (\Lambda_u + \omega I_{\mathbb{R}^{N_u}})\zeta_u + \mathcal{B}_u \mathbf{f}, \ \ \zeta_u(0) = \zeta_u^0, \\ \dfrac{d}{dt}\begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix} = \mathcal{A}_{s,\omega} \begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix} + \mathcal{B}_s \mathbf{f}, \ \begin{pmatrix} \mathbf{v}_s \\ \eta_{1,s} \\ \eta_{2,s} \end{pmatrix}(0) = \Pi_s \begin{pmatrix} \mathbf{v}^0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \end{cases} \tag{5.7.7}$$

where the matrix Λ_u is given by

$$\Lambda_u = [\Lambda_{i,j}]_{1\le i,j\le N_u}, \ \ \Lambda_{i,j} = \left\langle \mathcal{A}(P\mathbf{v}_i, \eta_{1,i}, \eta_{2,i})^\top, M^*(P\mathbf{\Phi}_j, \zeta_{1,j}, \zeta_{2,j})^\top \right\rangle_{\mathbf{Z},\mathbf{Z}'},$$

$$\zeta_u^0 = \left(\left\langle \begin{pmatrix} P\mathbf{v}^0 \\ 0 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0}\right)_{1\le j\le N_u} \quad \text{and} \quad \mathcal{B}_s = \Pi_s \mathcal{B}.$$

5.7.1 Stabilizability of $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$

In order to prove that the pair $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ is stabilizable, we must make two additional assumptions. Let us first consider the system

$$\begin{cases} \lambda\mathbf{\Phi} - \mathrm{div}\,\sigma(\mathbf{\Phi}, \psi) - (\mathbf{u}_s \cdot \nabla)\mathbf{\Phi} + (\nabla \mathbf{u}_s)^\top \mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathrm{div}\,\mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathbf{\Phi} = \zeta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d \setminus \Gamma_s, \ \ \sigma(\mathbf{\Phi}, \psi)\mathbf{n} + \mathbf{u}_s \cdot \mathbf{n}\mathbf{\Phi} = 0 \ \text{ on } \Gamma_n, \\ \lambda\zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{+,-}A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*\psi = 0 \ \text{ in } (0, \ell_s), \\ \lambda\zeta_2 + \alpha\Delta_s^2\zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^*\mathbf{\Phi} = -\gamma_s^+\psi + \gamma_s^-\psi \ \text{ in } (0, \ell_s), \\ \zeta_1(0) = \partial_x\zeta_1(0) = 0 \ \text{ and } \ \partial_x^2\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_2(\ell_s), \ \ \partial_x^3\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s). \end{cases} \tag{5.7.8}$$

Notice that the last three equations in (5.7.8) can be rewritten as

$$(\lambda I - A_s^*)\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \begin{pmatrix}\dfrac{1}{\alpha}(\Delta_s)^{-1}(A_1^*\mathbf{\Phi} - A_3^*\Psi)\\ A_2^*\mathbf{\Phi} - \gamma_s^{+,-}\Psi,\end{pmatrix} \tag{5.7.9}$$

where $(A_s^*, \mathcal{D}(A_s^*))$ is an unbounded operator in $H^2_{\{0\}}(0,\ell_s)\times L^2(0,\ell_s)$ defined by

$$\mathcal{D}(A_s^*) = E_s(0,\ell_s),\ \ A_s^* = \begin{pmatrix} 0 & -I + \dfrac{1}{\alpha}(\Delta_s)^{-1}\\ \alpha\Delta_s^2 & -\delta(\Delta_s^2)^{\frac{1}{2}}\end{pmatrix}.$$

Here,

$$\begin{aligned}E_s(0,\ell_s) = \Big\{(\zeta_1,\zeta_2)\in (H^4(0,\ell_s)\cap H^2_{\{0\}}(0,\ell_s))\times H^2_{\{0\}}(0,\ell_s)\mid \zeta_1{}''(\ell_s) = \frac{2\nu e}{\alpha}\zeta_2(\ell_s)\\ \text{and}\ \ \zeta_1{}'''(\ell_s) = \frac{2\nu e}{\alpha}\zeta_2'(\ell_s)\Big\}.\end{aligned}$$

We now proceed to state the two remaining assumptions.

$$\underline{\textbf{Assumption 2}}: \text{We assume that } -\omega\notin \operatorname{spec}(\mathcal{A}),\ 0\notin\operatorname{spec}(\mathcal{A}) \text{ and } \{\lambda\in\operatorname{spec}(A^*)\mid \Re\lambda\geq -\omega\}\cap\{\lambda\in\operatorname{spec}(A_s^*)\mid\Re\lambda\geq-\omega\}=\emptyset. \tag{A2}$$

For a given $\lambda\in\mathbb{C}$ satisfying $\Re\lambda\geq-\omega$, let us consider the following system:

$$\begin{cases}\lambda\mathbf{\Phi} - \operatorname{div}\sigma(\mathbf{\Phi},\psi) - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi} + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi} = 0 \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{\Phi} = 0\ \text{ in }\Omega,\\ \mathbf{\Phi} = 0\ \text{ on }\Gamma_d,\ \ \sigma(\mathbf{\Phi},\psi)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi} = 0\ \text{ on }\Gamma_n.\end{cases} \tag{5.7.10}$$

$$\underline{\textbf{Assumption 3}}: \text{We assume that if } (\lambda,\mathbf{\Phi},\psi) \text{ is solution to (5.7.10) and } \lambda(A_2^*\mathbf{\Phi} - \gamma_s^{+,-}\psi) = A_3^*\psi - A_1^*\mathbf{\Phi} \text{ holds, then } (\mathbf{\Phi},\psi) = (0,0). \tag{A3}$$

Theorem 5.7.1. *Let us suppose that Assumptions A1, A2 and A3 are satisfied and that $(w_i)_{1\leq i\leq N_c}$ is given by (5.7.3). Then, the pair $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ is stabilizable.*

Proof. From [BDDM07, Proposition 3.3, p.492], we know that the pair $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ is stabilizable, if and only if,

$$\ker(\lambda I - \mathcal{A}^*)\cap\ker(\mathcal{B}^*) = \{0\}\ \text{ for all } \lambda\in\mathbb{C} \text{ such that } \Re\lambda\geq-\omega.$$

Let us assume that $M^*(P\mathbf{\Phi},\zeta_1,\zeta_2)^\top\in\ker(\lambda I - \mathcal{A}^*)\cap\ker(\mathcal{B}^*)$. By setting $(I-P)\mathbf{\Phi} = -\nabla N_s\zeta_2$, from Theorem 5.4.5 we deduce that there exists $\psi\in H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)$ such that the quadruplet $(\mathbf{\Phi},\psi,\zeta_1,\zeta_2)$ is solution to (5.7.8). On the other hand, since

$$\mathcal{B}^*M^*\begin{pmatrix}P\mathbf{\Phi}\\ \zeta_1\\ \zeta_2\end{pmatrix} = 0,$$

then

$$\left(\int_0^{\ell_s} w_i\zeta_2\right)_{1\leq i\leq N_c} = 0. \tag{5.7.11}$$

In what follows, we will demonstrate that $(\mathbf{\Phi},\psi,\zeta_1,\zeta_2) = (\mathbf{0},0,0,0)$. Let us first notice that thanks to the fact that the family $(w_i)_{1\leq i\leq N_c}$ satisfies the condition (5.7.3), the equality (5.7.11) implies that $\zeta_2 = 0$. Now, we will distinguish two cases:

• **Case 1:** $\lambda \notin \mathrm{spec}(A^*)$. Under this assumption, we have that $(\mathbf{\Phi}, \psi) = (\mathbf{0}, 0)$. Next, using this fact in the fourth equation in (5.7.8), we obtain that $\zeta_1 = 0$.

• **Case 2:** $\lambda \in \mathrm{spec}(A^*)$. The Assumption A2 implies that $\lambda \notin \mathrm{spec}(A_s^*)$, and thus, the last three equations in (5.7.8), together with identity (5.7.9), allow us to deduce that

$$\lambda(A_2^*\mathbf{\Phi} - \gamma_s^{+,-}\psi) = A_3^*\psi - A_1^*\mathbf{\Phi}.$$

From Assumption A3, it follows that $(\mathbf{\Phi}, \psi) = (0, 0)$. Finally, using the fourth equation in (5.7.8), we infer that $\zeta_1 = 0$. This completes the proof. □

5.7.2 Feedback control law

We are now interested in finding a finite-dimensional feedback law that allows us to stabilize the linearized fluid-structure system (5.2.11). Based on the results of the previous subsection, it is sufficient to find a feedback law that stabilizes system (5.9.71). To achieve this, let us note that, since the pair $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$ is stabilizable and $-(\Lambda_u + \omega I_{\mathbb{R}^{N_u}})$ is stable, it follows from [KR09, Theorem 3] that the algebraic Riccati equation

$$\begin{aligned} &Q_u \in \mathcal{L}(\mathbb{R}^{N_u}), \quad Q_u = Q_u^\top > 0, \\ &(\Lambda_u^\top + \omega I_{\mathbb{R}^{N_u}})Q_u + Q_u(\Lambda_u + \omega I_{\mathbb{R}_{N_u}}) - Q_u\mathcal{B}_u\mathcal{B}_u^\top Q_u = 0 \end{aligned} \tag{5.7.12}$$

admits a unique solution. Consequently, the operator $K_u = (K_u^{i,j})_{1\leq i\leq N_c, 1\leq j\leq N_u}$ defined by $K_u = -\mathcal{B}_u^\top Q_u$, serves as a feedback law for the pair $(\Lambda_u + \omega I_{\mathbb{R}^{N_u}}, \mathcal{B}_u)$. Let us now introduce the operator $\mathcal{K}_p \in \mathcal{L}(\mathbf{Z}_0, \mathbb{R}^{N_c})$ defined by

$$\mathcal{K}_p \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \left(\sum_{j=1}^{N_u} K_u^{i,j} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1\leq i\leq N_c}. \tag{5.7.13}$$

As a consequence of the fact that the stability of $\mathcal{A} + \omega I + \mathcal{B}\mathcal{K}_p$ is equivalent to the stability of $\Lambda_u + \omega I_{\mathbb{R}^{N_u}} + \mathcal{B}_u K_u$, we obtain the following result, whose proof can be adapted from [Ndi16, Theorem 2.7.7, p. 89].

Theorem 5.7.2. *Under Assumptions A1, A2 and A3, the operator $\mathcal{K}_p$ introduced in (5.7.13) provides a feedback law for the pair $(\mathcal{A} + \omega I, \mathcal{B})$. Moreover, the operator $\mathcal{A} + \omega I + \mathcal{B}\mathcal{K}_p$, with domain $D(\mathcal{A} + \omega I + \mathcal{B}\mathcal{K}_p) = D(\mathcal{A})$, is the generator of an exponentially stable analytic semigroup on $\mathbf{Z}$.*

Let us consider the feedback law $\mathcal{K} \in \mathcal{L}(\mathbf{H}_0, \mathbb{R}^{N_c})$ defined by

$$\mathcal{K} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \left(\mathcal{K}_i \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} \right)_{1\leq i\leq N_c}, \tag{5.7.14}$$

where

$$\mathcal{K}_i \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \sum_{j=1}^{N_u} K_u^{i,j} \left\langle \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{H}_0}, \tag{5.7.15}$$

for each $1 \leq i \leq N_c$. From Proposition 5.7.1, we deduce the following result that allows us to establish the link between the feedback laws (5.7.13) and (5.7.14).

Proposition 5.7.4. *For all* $(P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{Z}_0$*, if we choose* $(I-P)\mathbf{v} = \nabla N_{\mathrm{div}} A_2\eta_2 - \nabla N_s \eta_2 \in \mathbf{L}^2(\Omega)$*, we obtain*

$$\mathcal{K}_p \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{K} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}. \tag{5.7.16}$$

5.8 Stabilization of the nonhomogeneous linearized system

In this section, we will prove that the feedback law $\mathcal{K}$ defined in (5.7.14) stabilizes the nonhomogeneous linear system

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, q) + (\mathbf{u}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega \mathbf{v} = \mathbf{F}_f \ \text{ in } Q^\infty, \\ \operatorname{div} \mathbf{v} = A_3\eta_1 + \operatorname{div} \mathbf{G}_{\mathrm{div}} \ \text{ in } Q^\infty, \\ \mathbf{v} = \mathbf{g}_p \ \text{ on } \Sigma_i^\infty, \ \ \mathbf{v} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma_r^\infty \cup \Sigma_w^\infty, \ \ \sigma(\mathbf{v}, q)\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty, \\ \partial_t \eta_1 - \eta_2 - \omega\eta_1 = 0 \ \text{ in } (0, \infty) \times (0, \ell_s), \\ \partial_t \eta_2 + \alpha \Delta^2 \eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}} \eta_2 - A_4\eta_1 - \omega\eta_2 = -\gamma_s^+ q + \gamma_s^- q + F_s \\ \qquad\qquad + \sum_{i=1}^{N_c} [\mathcal{K}(\mathbf{v}, \eta_1, \eta_2)^\top] w_i \ \text{ in } (0, \infty) \times (0, \ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ on } (0, \infty) \times \{0\} \ \text{ and } \ \partial_{x_1}^2 \eta_1 = 0, \partial_{x_1}^3 \eta_1 = 0 \ \text{ on } (0, \infty) \times \{\ell_s\}, \\ \eta_1(0) = 0 \ \text{ and } \ \eta_2(0) = \eta_2^0 \ \text{ in } (0, \ell_s). \end{cases} \tag{5.8.1}$$

Before stating the main result of this section, we begin by recalling two useful results established in Section 3.4 of Chapter 3 concerning the lifiting of the Dirichlet boundary data $\mathbf{g}_p$ and the divergence constraint data $\mathbf{G}_{\mathrm{div}}$.

We first state the result related to the lifting of the non-homogeneous Dirichlet data $\mathbf{g}_p$ in system (5.8.1). Let us consider the following system:

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{z}(t), \pi(t)) = 0 \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{z}(t) = 0 \ \text{ in } \Omega, \\ \mathbf{z}(t) = \mathbf{g}_p(t) \ \text{ on } \Gamma_i, \ \ \mathbf{z}(t) = 0 \ \text{ on } \Gamma_r \cup \Gamma_s \cup \Gamma_w, \\ \sigma(\mathbf{z}(t), \pi(t))\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{5.8.2}$$

Proposition 5.8.1. *Let* $\alpha \in (0, \alpha^*)$*,* $\delta \in (\delta^*, 1)$*. For all* $\mathbf{g}_p \in H^1_{\{0\}}(0, \infty; \mathbf{H}(\Gamma_i))$*, system* (5.8.2) *admits unique solution* $(\mathbf{z}, \pi) \in H^1(0, \infty; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \times H^1(0, \infty; H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))$*. Moreover, there exists a positive constant* $C_{\alpha,\delta}$*, such that*

$$\|\mathbf{z}\|_{H^1(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))} + \|\pi\|_{H^1(0,\infty;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))} \le C_{\alpha,\delta} \|\mathbf{g}_p\|_{H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))}. \tag{5.8.3}$$

Before presenting the result concerning the lifting for the divergence constraint data $\mathbf{G}_{\mathrm{div}}$, we introduce some preliminary notation. We consider the following decomposition of $\mathbf{G}_{\mathrm{div}}$:

$$\mathbf{G}_{\mathrm{div}} = \widetilde{\theta}\mathbf{G}_{\mathrm{div}} + (1 - \widetilde{\theta})\mathbf{G}_{\mathrm{div}}, \tag{5.8.4}$$

where $\widetilde{\theta}$ is the cut-off function introduced in (3.4.7).

Let $(\mathbf{w}_L(t), p_L(t))$ be the solution of system

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{w}_L(t), p_L(t)) = 0 \ \text{ in } \Omega, \ \ \operatorname{div} \mathbf{w}_L(t) = \operatorname{div}(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}) \ \text{ in } \Omega, \\ \mathbf{w}_L(t) = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{w}_L(t), p_L(t))\mathbf{n} = 0 \ \text{ on } \Gamma_n, \end{cases} \tag{5.8.5}$$

with $\widetilde{\theta}\mathbf{G}_{\mathrm{div}}$ satisfying
$$\mathrm{div}(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}) \in L^2(0,\infty; H^{\frac{1}{2}+\alpha,1}(\Omega)) \tag{5.8.6}$$
and
$$\widetilde{\theta}\mathbf{G}_{\mathrm{div}} \in H^1(0,\infty; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)). \tag{5.8.7}$$
Let $\mathbf{w}_R := (1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}$, with $(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}$ satisfying
$$\begin{aligned}
&(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}} \in L^2(0,\infty; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)),\\
&(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}} = 0 \ \text{ in } \Omega_2,\\
&(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}} = 0 \ \text{ on } \Sigma_d^\infty, \ \ \varepsilon((1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}})\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty.
\end{aligned} \tag{5.8.8}$$
Let us now set
$$\mathbf{w} := \mathbf{w}_L + \mathbf{w}_R. \tag{5.8.9}$$

Proposition 5.8.2. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*, 1)$. Under the assumptions (5.8.6) and (5.8.7), along with condition (5.8.8), the function $\mathbf{w}$ defined in (5.8.9) belongs to $L^2(0,\infty; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$. Moreover, there exists a constant $C_{\alpha,\delta} > 0$, such that*
$$\begin{aligned}
&\|\mathbf{w}\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))} + \|\mathbf{w}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\\
&\le C_{\alpha,\delta}\Big(\Big\|\mathrm{div}\left(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\right)\Big\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\\
&\qquad\qquad + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\Big).
\end{aligned} \tag{5.8.10}$$

We are now in position to state the main result of this section.

Theorem 5.8.1. *Let $\alpha \in (0,\alpha^*)$, $\delta \in (\delta^*, 1)$. We suppose that Assumptions A1, A2 and A3 are satisfied. Let us assume that $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H^1_{\{0\}}(0,\ell_s)$, $\mathbf{F}_f \in L^2(0,\infty; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$, $\mathbf{g}_p \in H^1_{\{0\}}(0,\infty; \mathbf{H}(\Gamma_i))$ and $F_s \in L^2(0,\infty; L^2(0,\ell_s))$. Let us assume that $\mathbf{G}_{\mathrm{div}}$ satisfies assumptions (5.8.6) and (5.8.7), along with condition (5.8.8) and $\mathbf{G}_{\mathrm{div}}|_{t=0} = 0$, and let $\mathbf{w}$ denotes its lifting defined as in (5.8.9). We assume in addition the following conditions:*
$$\begin{cases}
\mathbf{v}_0 = 0 \ \text{ on } \Gamma_i, \ \ \mathbf{v}_0 = 0 \ \text{ on } \Gamma_r \cup \Gamma_w,\\
\mathbf{v}_0 = \eta_2(0,\cdot)\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \mathrm{div}\,\mathbf{v}_0 = 0 \ \text{ in } \Omega,\\
(P\mathbf{v}_0, 0, \eta_2^0) \in [\mathcal{D}(\mathcal{A}), \mathbf{Z}]_{1/2}, \ \ P\mathbf{v}_0 \in [\mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)), \mathbf{V}^0_{n,\Gamma_d}(\Omega)]_{1/2}.
\end{cases} \tag{5.8.11}$$

Then, system (5.8.1) admits a unique solution $(\mathbf{v}, p, \eta_1, \eta_2)$ belonging to $L^2(0,\infty; \mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty; \mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \times L^2(0,\infty; H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)) \times H^{4,2}((0,\infty)\times(0,\ell_s)) \times H^{2,1}((0,\infty)\times(0,\ell_s))$, satisfying the estimate
$$\begin{aligned}
&\|\mathbf{v}\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))} + \|\mathbf{v}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|p\|_{L^2(0,\infty;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega))}\\
&+ \|\zeta_1\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\zeta_2\|_{H^{2,1}((0,\infty)\times(0,\ell_s))}\\
&\le C_L\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\partial_t\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\eta_2^0\|_{H^1(0,\ell_s)}\\
&\qquad + \|\mathbf{F}_\mathbf{f}\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|F_s\|_{L^2((0,\infty)\times(0,\ell_s))}\\
&\qquad + \Big\|\mathrm{div}\left(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\right)\Big\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\\
&\qquad + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\Big),
\end{aligned} \tag{5.8.12}$$

where C_L is a positive constant.

Proof. We split the proof in five steps.

• **Step 1**. *Reformulation of the system* (5.8.1).

Let $(\mathbf{z}(t), \pi(t))$ be the solution to system (5.8.2). Let $\mathbf{w}$ be the vector field given in (5.8.9). We set $\widetilde{\mathbf{v}} = \mathbf{v} - \mathbf{w} - \mathbf{z}$ and $\widetilde{p} = p - p_L - \pi$. The couple $(\widetilde{\mathbf{v}}, \widetilde{p})$ satisfies

$$\begin{cases} \partial_t \widetilde{\mathbf{v}} - \operatorname{div} \sigma(\widetilde{\mathbf{v}}, \widetilde{p}) + (\mathbf{u}_s \cdot \nabla)\widetilde{\mathbf{v}} + (\widetilde{\mathbf{v}} \cdot \nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 - \omega\widetilde{\mathbf{v}} = \widetilde{\mathbf{F}}_f \ \text{ in } Q^\infty, \\ \operatorname{div} \widetilde{\mathbf{v}} = A_3\eta_1 \ \text{ in } Q^\infty, \\ \widetilde{\mathbf{v}} = 0 \ \text{ on } \Sigma_i^\infty, \ \widetilde{\mathbf{v}} = \eta_2 \vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \widetilde{\mathbf{v}} = 0 \ \text{ on } \Sigma_r^\infty \cup \Sigma_w^\infty, \ \sigma(\widetilde{\mathbf{v}}, \widetilde{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty, \\ \partial_t \eta_1 - \eta_2 - \omega\eta_1 = 0 \ \text{ in } (0,\infty) \times (0, \ell_s), \\ \partial_t \eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 - A_4\eta_1 - \omega\eta_2 = -\gamma_s^+\widetilde{p} + \gamma_s^-\widetilde{p} + \widetilde{F}_s \ \text{ in } (0,\infty) \times (0, \ell_s), \\ \eta_1 = 0, \partial_{x_1}\eta_1 = 0 \ \text{ on } (0,\infty) \times \{0\} \ \text{and} \ \partial_{x_1}^2\eta_1 = 0, \partial_{x_1}^3\eta_1 = 0 \ \text{ on } (0,\infty) \times \{\ell_s\}, \\ \eta_1(0) = 0 \ \text{ and } \ \eta_2(0) = \eta_2^0 \ \text{ in } (0, \ell_s), \end{cases} \tag{5.8.13}$$

where

$$\begin{aligned} &\widetilde{\mathbf{F}} = \mathbf{F}_f - \partial_t \mathbf{w} - \partial_t \mathbf{z} + 2\nu \operatorname{div} \varepsilon(\mathbf{w}) - (\mathbf{u}_s \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{u}_s + \omega\mathbf{w} + \lambda_f \mathbf{z} + \omega\mathbf{z}, \\ &\widetilde{F}_s = F_s - \gamma_s^+\pi + \gamma_s^-\pi + \sum_{i=1}^{N_c} \mathcal{K}_i(\mathbf{w} + \mathbf{z}, \eta_1, \eta_2)^\top w_i. \end{aligned} \tag{5.8.14}$$

Thus, following the same idea used to prove Theorem 5.3.5 we deduce that the solution $(\widetilde{\mathbf{v}}, \widetilde{p}, \eta_1, \eta_2)$ of (5.8.13) satisfies

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} P\widetilde{\mathbf{v}} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} P\widetilde{\mathbf{v}} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} P\widetilde{\mathbf{F}} \\ 0 \\ \widetilde{H} \end{pmatrix}, \quad \begin{pmatrix} P\widetilde{\mathbf{v}}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}_0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \\ (I-P)\widetilde{\mathbf{v}} = \nabla N_{\mathrm{div}} A_3\eta_1 - \nabla N_s \eta_2, \end{cases} \tag{5.8.15}$$

where $\widetilde{H} = (I + \gamma_s^{+,-}N_s)^{-1}\left(\widetilde{F}_s - \gamma_s^{+,-}N_p\widetilde{\mathbf{F}}\right)$.

• **Step 2**. *Regularity of* $(P\widetilde{\mathbf{v}}, \eta_1, \eta_2)$.

Let us first observe that, thanks to Propositions 5.8.1 and 5.8.2, Assumption A1 and [GS91, Proposition B.1], we obtain

$$\begin{aligned} &\|P\widetilde{\mathbf{F}}_f\|_{L^2(0,\infty;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega))} \le C\|\widetilde{\mathbf{F}}_f\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ &\le C\Big(\|\mathbf{F}_f\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \left\|\operatorname{div}\left(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\right)\right\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} \\ &+ \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \left\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\right\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\ &+ \|\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\partial_t \mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))}\Big) \end{aligned} \tag{5.8.16}$$

and

$$\begin{aligned}
&\|\widetilde{F}_s\|_{L^2(0,\infty;L^2(0,\ell_s))} \le C\Big(\|\mathbf{F}_f\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \Big\|\mathrm{div}\,\Big(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\Big)\Big\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} \\
&+ \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&+ \|\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\partial_t\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\widetilde{F}_s\|_{L^2(0,\infty;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{5.8.17}$$

Since $\mathcal{A}+\omega I+\mathcal{BK}$ is the infinitesimal generator of an exponentially stable analytic semigroup on $\mathbf{Z}$ (see Theorem 5.7.2), $(P\widetilde{\mathbf{F}},0,\widetilde{F}_s)^\top \in \mathbf{Z}$ (see estimates (5.8.16) and (5.8.17)) and $(P\mathbf{v}_0,0,\eta_2^0)\in[\mathcal{D}(\mathcal{A}),\mathbf{Z}]_{1/2}$, the maximal regularity result [BDDM07, Theorem 3.1, p. 143] implies that $(P\widetilde{\mathbf{v}},\eta_1,\eta_2)$ belongs to $L^2(0,\infty;\mathcal{D}(\mathcal{A}))\cap H^1(0,\infty;\mathbf{Z})$ and satisfies

$$\begin{aligned}
&\|P\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))} + \|\eta_1\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\eta_2\|_{H^{2,1}((0,\infty)\times(0,\ell_s))} \\
&\le C\left(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|P\widetilde{\mathbf{F}}\|_{L^2(0,\infty;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|\widetilde{F}_s\|_{L^2(0,\infty;L^2(0,\ell_s))}\right).
\end{aligned}$$

Furthermore, using again estimates (5.8.16) and (5.8.17) to bound the right-hand side in the previous estimate, we obtain

$$\begin{aligned}
&\|P\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))} + \|\eta_1\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\eta_2\|_{H^{2,1}((0,\infty)\times(0,\ell_s))} \\
&\le C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\mathbf{F}_f\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \Big\|\mathrm{div}\,\Big(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\Big)\Big\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} \\
&+ \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&+ \|\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\partial_t\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\widetilde{F}_s\|_{L^2(0,\infty;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{5.8.18}$$

• **Step 3**. *Estimate of $\widetilde{\mathbf{v}}$ in the $H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$−norm.*

Using the fact that $\nabla N_{\mathrm{div}}A_3\eta_1 - \nabla N_s\eta_2 = (I-P)L(\eta_2,A_3\eta_1)$, where $L\in\mathcal{L}(H^2_{\{0\}}(0,\ell_s)\times H^{\frac{1}{2}+\alpha,1}(\Omega),\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))$ is the lifting operator defined in (5.3.23), along with Theorem 5.3.1 (ii) and the transposition method, we have that $L\in\mathcal{L}(L^2(0,\ell_s)\times H^{\frac{1}{2}+\alpha,1}(\Omega),\mathbf{L}^2(\Omega))$. Then, there exists a positive constant C such that

$$\begin{aligned}
\|(I-P)\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} &= \|(I-P)L(\eta_2,A_3\eta_1)\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\|(I-P)L(\eta_2,A_3\eta_1)\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} \\
&\le C(\|\eta_1\|_{H^{4,2}((0,\infty)\times(0,\ell_s))} + \|\eta_2\|_{H^{2,1}((0,\infty)\times(0,\ell_s))}).
\end{aligned} \tag{5.8.19}$$

Then, after combining the estimates (5.8.18) and (5.8.19), we get

$$\begin{aligned}
&\|\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} \le \|P\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} + \|(I-P)\widetilde{\mathbf{v}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\mathbf{F}_f\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0})} + \Big\|\mathrm{div}\,\Big(\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\Big)\Big\|_{L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))} \\
&+ \|\widetilde{\theta}\mathbf{G}_{\mathrm{div}}\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \Big\|(1-\widetilde{\theta})\mathbf{G}_{\mathrm{div}}\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \\
&+ \|\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\partial_t\mathbf{g}_p\|_{L^2(0,\infty;\mathbf{H}(\Gamma_i))} + \|\widetilde{F}_s\|_{L^2(0,\infty;L^2(0,\ell_s))}\Big).
\end{aligned} \tag{5.8.20}$$

- **Step 4**. *Estimates of* $\widetilde{\mathbf{v}}$ *in the* $\Big(L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))\Big)$*−norm and* $\widetilde{p}$ *in the* $L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega))$*−norm.*

We will show that

$$\widetilde{\mathbf{v}}\in L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_{\delta}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)) \ \text{ and } \ \widetilde{p}\in L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_{\delta}(\Omega)). \tag{5.8.21}$$

Let us first consider the system

$$\begin{cases}\lambda_f\widetilde{\mathbf{v}}-\operatorname{div}\sigma(\widetilde{\mathbf{v}},\widetilde{p})+(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}+(\widetilde{\mathbf{v}}\cdot\nabla)\mathbf{u}_s=A_1\eta_1+A_2\eta_2+\omega\widetilde{\mathbf{v}}+\mathbf{F}-\partial_t\widetilde{\mathbf{v}}+\lambda_f\widetilde{\mathbf{v}} \ \text{ in } \Omega,\\ \operatorname{div}\widetilde{\mathbf{v}}=A_3\eta_1 \ \text{ in } \Omega,\\ \widetilde{\mathbf{v}}=\eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \widetilde{\mathbf{v}}=0 \ \text{ on } \Gamma_i\cup\Gamma_r\cup\Gamma_w, \ \ \sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n}=0 \ \text{ on } \Gamma_n,\end{cases}$$

where λ_f is the parameter appearing in Theorem 5.3.1.

Since $A_1\eta_1, A_2\eta_2, \mathbf{F}, \widetilde{\mathbf{v}}, \partial_t\widetilde{\mathbf{v}}\in L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))$ we deduce that

$$A_1\eta_1+A_2\eta_2+\omega\widetilde{\mathbf{v}}+\mathbf{F}-\partial_t\widetilde{\mathbf{v}}+\lambda_f\widetilde{\mathbf{v}}\in L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega)).$$

Then, since in addition $A_3\eta_1\in L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}(\Omega))$ and $\eta_2\in H^{2,1}((0,\infty)\times(0,\ell_s))$, Theorem 5.3.1 (i) implies that

$$\widetilde{\mathbf{v}}\in L^2(0,\infty;\mathbf{H}^1(\Omega)) \ \text{ and } \ \widetilde{p}\in L^2(0,\infty;L^2(\Omega)). \tag{5.8.22}$$

Let us set $(\widetilde{\mathbf{v}}_1,\widetilde{p}_1):=(\Psi\widetilde{\mathbf{v}},\Psi\widetilde{p})$ and $(\widetilde{\mathbf{v}}_2,\widetilde{p}_2):=((1-\Psi)\widetilde{\mathbf{v}},(1-\Psi)\widetilde{p})$. To show (5.8.21), it is suffices to prove that

$$\widetilde{\mathbf{v}}_1\in L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)) \ \text{ and } \ \widetilde{p}_1\in L^2(0,\infty;H^{\frac{1}{2}+\alpha}(\Omega)) \tag{5.8.23}$$

and

$$\widetilde{\mathbf{v}}_2\in L^2(0,\infty;\mathbf{H}^2_{\delta}(\Omega))\cap H^1(0,\infty;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{p}_2\in L^2(0,\infty;H^1_{\delta}(\Omega)). \tag{5.8.24}$$

(i) Proof of (5.8.23). Firstly, since $\widetilde{\mathbf{v}}\in H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$, then $\widetilde{\mathbf{v}}_1\in H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$. Let us show that

$$\widetilde{\mathbf{v}}_1\in L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)) \ \text{ and } \ \widetilde{p}_1\in L^2(0,\infty;H^{\frac{1}{2}+\alpha}(\Omega)). \tag{5.8.25}$$

We first set

$$\overline{\mathbf{F}}=A_1\eta_1+A_2\eta_2+\omega\widetilde{\mathbf{v}}+\mathbf{F}+(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}. \tag{5.8.26}$$

Let us observe that the couple $(\widetilde{\mathbf{v}}_1,\widetilde{p}_1)$ solves

$$\begin{cases}-\operatorname{div}\sigma(\widetilde{\mathbf{v}}_1,\widetilde{p}_1)=\overline{\mathbf{F}}_1-\partial_t\widetilde{\mathbf{v}}_1-(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_1-(\widetilde{\mathbf{v}}_1\cdot\nabla)\mathbf{u}_s \ \text{ in } Q^\infty,\\ \operatorname{div}\widetilde{\mathbf{v}}_1=h_1 \ \text{ in } Q^\infty,\\ \widetilde{\mathbf{v}}_1=\Psi\eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \ \widetilde{\mathbf{v}}_1=0 \ \text{ on } \Sigma_i^\infty\cup\Sigma_r^\infty\cup\Sigma_w^\infty\cup\Sigma_n^\infty,\end{cases}$$

where

$$\overline{\mathbf{F}}_1=\Psi\overline{\mathbf{F}}+\widetilde{p}\nabla\Psi-\nu\left[\widetilde{\mathbf{v}}\Delta\Psi+2(\nabla\Psi\cdot\nabla)\widetilde{\mathbf{v}}-(\operatorname{div}\widetilde{\mathbf{v}})\nabla\Psi+\nabla(\widetilde{\mathbf{v}}\cdot\nabla\Psi)\right]+[(\nabla\Psi)\widetilde{\mathbf{v}}^\top]\mathbf{u}_s$$

and

$$h_1=\Psi(A_3\eta_1)+\nabla\Psi\cdot\widetilde{\mathbf{v}}.$$

We claim that $(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_1$ and $(\widetilde{\mathbf{v}}_1\cdot\nabla)\mathbf{u}_s$ belongs to $L^2(0,\infty;\mathbf{L}^2(\Omega))$. Indeed, using

Assumption A1, the continuous embedding $\mathbf{H}^{\frac{3}{2}+\alpha}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ and the fact that $\widetilde{\mathbf{v}}_1 \in L^2(0,\infty;\mathbf{H}^1(\Omega))$, we obtain

$$\int_0^\infty \|(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_1\|^2_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{u}_s\|^2_{\mathbf{L}^\infty(\Omega)} \int_0^\infty \|\nabla\widetilde{\mathbf{v}}_1\|^2_{\mathbf{L}^2(\Omega)} < \infty, \tag{5.8.27}$$

from where we deduce that $(\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_1 \in L^2(0,\infty;\mathbf{L}^2(\Omega))$. On the other hand, using again the Assumption A1, the Cauchy-Schwarz inequality, the continuous embeddings $\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$ and $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$, together with the fact that $\widetilde{\mathbf{v}}_1 \in L^2(0,\infty;\mathbf{H}^1(\Omega))$, we obtain

$$\begin{aligned}\int_0^\infty \|(\widetilde{\mathbf{v}}_1\cdot\nabla)\mathbf{u}_s\|^2_{\mathbf{L}^2(\Omega)} &\leq \int_0^\infty \|\nabla\mathbf{u}_s\|^2_{\mathbf{L}^4(\Omega)}\|\widetilde{\mathbf{v}}_1\|^2_{\mathbf{L}^4(\Omega)}\\ &\leq C\|\nabla\mathbf{u}_s\|^2_{\mathbf{H}^{\frac{1}{2}+\alpha}(\Omega)} \int_0^\infty \|\widetilde{\mathbf{v}}_1\|^2_{\mathbf{H}^1(\Omega)}\\ &< \infty,\end{aligned} \tag{5.8.28}$$

and thus, $(\widetilde{\mathbf{v}}_1\cdot\nabla)\mathbf{u}_s \in L^2(0,\infty;\mathbf{L}^2(\Omega))$.

Then, using (5.8.22), the fact that $\widetilde{\mathbf{v}}_1 \in H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$ along with estimates (5.8.27) and (5.8.28), we obtain that

$$\overline{\mathbf{F}}_1 - \partial_t\widetilde{\mathbf{v}}_1 - (\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_1 - (\widetilde{\mathbf{v}}_1\cdot\nabla)\mathbf{u}_s \in L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)).$$

On the other hand, since $\Psi(A_3\eta_1) \in L^2(0,\infty;H^{\frac{1}{2}+\alpha}(\Omega))$ and $\widetilde{\mathbf{v}}_1 \in L^2(0,\infty;\mathbf{H}^1(\Omega))$, we get

$$h_1 \in L^2(0,\infty;H^{\frac{1}{2}+\alpha}(\Omega)).$$

Finally, after applying [Dau89, Theorem 5.5(a)] and [BR, Theorem 3.2 and Corollary 3.3], we obtain (5.8.25).

(*ii*) Proof of (5.8.24). Let us first observe that the couple $(\widetilde{\mathbf{v}}_2,\widetilde{p}_2)$ solves

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{v}}_2,\widetilde{p}_2) = \overline{\mathbf{F}}_2 - \partial_t\widetilde{\mathbf{v}}_2 - (\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_2 - (\widetilde{\mathbf{v}}_2\cdot\nabla)\mathbf{u}_s \ \text{ in } Q^\infty,\\ \operatorname{div}\widetilde{\mathbf{v}}_2 = h_2 \ \text{ in } Q^\infty,\\ \widetilde{\mathbf{v}}_2 = (1-\Psi)\eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty, \ \ \widetilde{\mathbf{v}}_1 = 0 \ \text{ on } \Sigma_d^\infty\setminus\Sigma_s^\infty, \ \ \sigma(\widetilde{\mathbf{v}}_1,\widetilde{p}_2)\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty,\end{cases}$$

where

$$\overline{\mathbf{F}}_2 = (1-\Psi)\overline{\mathbf{F}} - \widetilde{p}\nabla\Psi + \nu\left[\widetilde{\mathbf{v}}\Delta\Psi + 2(\nabla\Psi\cdot\nabla)\widetilde{\mathbf{v}} - (\operatorname{div}\widetilde{\mathbf{v}})\nabla\Psi + \nabla(\widetilde{\mathbf{v}}\cdot\nabla\Psi)\right] + [(\nabla\Psi)\widetilde{\mathbf{v}}^\top]\mathbf{u}_s$$

and

$$h_2 = (1-\Psi)A_3\eta_1 - \nabla\Psi\cdot\widetilde{\mathbf{v}}.$$

We will now construct two lifting functions: one corresponding to the boundary data $(1-\Psi)\eta_2\vec{\mathbf{e}}_2$, and another corresponding to the divergence data h_2.

- Lifting for the boundary data. Let us consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{w}}_1,\widetilde{\pi}_1) = 0, \ \ \operatorname{div}\widetilde{\mathbf{w}}_1 = 0 \ \text{ in } \Omega,\\ \widetilde{\mathbf{w}}_1 = (1-\Psi)\eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \ \ \widetilde{\mathbf{w}}_1 = 0 \ \text{ on } \Gamma_d\setminus\Gamma_s \ \ \sigma(\widetilde{\mathbf{w}}_1,\widetilde{\pi}_1)\mathbf{n} = 0 \ \text{ on } \Gamma_n.\end{cases} \tag{5.8.29}$$

Since $\eta_2 \in H^{2,1}((0,\infty)\times(0,\ell_s))$, thanks to [MR10, Theorem 9.4.5] and the transposition method, we deduce that

$$\widetilde{\mathbf{w}}_1 \in L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi}_1 \in L^2(0,\infty;H^1_\delta(\Omega)). \tag{5.8.30}$$

- Lifting for the divergence data. Let us consider the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{w}}_2,\widetilde{\pi}_2) = 0, \ \operatorname{div}\widetilde{\mathbf{w}}_2 = h_2 \ \text{ in } \Omega, \\ \widetilde{\mathbf{w}}_2 = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\widetilde{\mathbf{w}}_2,\widetilde{\pi}_2)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{5.8.31}$$

Since $h_2 \in L^2(0,\infty;H^1(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}_{\Gamma_d}(\Omega))$, from [MR10, Theorem 9.4.5] and Lemma 3.6.1 we deduce that

$$\widetilde{\mathbf{w}}_2 \in L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi}_2 \in L^2(0,\infty;H^1_\delta(\Omega)). \tag{5.8.32}$$

Setting $\widetilde{\mathbf{w}} := \widetilde{\mathbf{w}}_1 + \widetilde{\mathbf{w}}_2$ and $\widetilde{\pi} := \widetilde{\pi}_1 + \widetilde{\pi}_2$, we deduce from (5.8.29) and (5.8.31) that

$$\widetilde{\mathbf{w}} \in L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{\pi} \in L^2(0,\infty;H^1_\delta(\Omega)). \tag{5.8.33}$$

After setting $\widetilde{\mathbf{v}}_2 = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$ and $\widetilde{p}_2 = \widetilde{\pi} + \widetilde{q}$, we observe that $(\widetilde{\mathbf{z}},\widetilde{q})$ solves

$$\begin{cases} \partial_t\widetilde{\mathbf{z}} - \operatorname{div}\sigma(\widetilde{\mathbf{z}},\widetilde{q}) = \overline{\mathbf{F}}_2 - \partial_t\widetilde{\mathbf{v}}_2 - (\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_2 - (\widetilde{\mathbf{v}}_2\cdot\nabla)\mathbf{u}_s - \partial_t\widetilde{\mathbf{w}}, \ \operatorname{div}\widetilde{\mathbf{z}} = 0 \ \text{ in } Q^\infty, \\ \widetilde{\mathbf{z}} = 0 \ \text{ on } \Sigma^\infty_d, \ \ \sigma(\widetilde{\mathbf{z}},\widetilde{q})\mathbf{n} = 0 \ \text{ on } \Sigma^\infty_n. \end{cases}$$

After using (5.8.22) and applying a similar argument to that used in (5.8.27) and (5.8.28) to estimate the convective terms, we deduce that

$$\overline{\mathbf{F}}_2 - \partial_t\widetilde{\mathbf{v}}_2 - (\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_2 - (\widetilde{\mathbf{v}}_2\cdot\nabla)\mathbf{u}_s - \partial_t\widetilde{\mathbf{w}} \in L^2(0,\infty;\mathbf{L}^2(\Omega)).$$

Since the semigroup generated by the Stokes operator $(A_0;\mathbf{V}^0_{n,\Gamma_d}(\Omega))$ on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is analytic and exponentially stable, the maximal regularity result [BDDM07, Theorem 3.1(i), p. 143], together with the constraint $(I-P)\widetilde{\mathbf{z}} = 0$, implies in particular that

$$\widetilde{\mathbf{z}} \in H^1(0,\infty;\mathbf{L}^2(\Omega)). \tag{5.8.34}$$

Then, from [MR10, Theorem 9.4.5] we deduce that the system

$$\begin{cases} -\operatorname{div}\sigma(\widetilde{\mathbf{z}},\widetilde{q}) = \overline{\mathbf{F}}_2 - \partial_t\widetilde{\mathbf{v}}_2 - (\mathbf{u}_s\cdot\nabla)\widetilde{\mathbf{v}}_2 - (\widetilde{\mathbf{v}}_2\cdot\nabla)\mathbf{u}_s - \partial_t\widetilde{\mathbf{w}} - \partial_t\widetilde{\mathbf{z}}, \ \operatorname{div}\widetilde{\mathbf{z}} = 0 \ \text{ in } Q^\infty, \\ \widetilde{\mathbf{z}} = 0 \ \text{ on } \Sigma^\infty_d, \ \ \sigma(\widetilde{\mathbf{z}},\widetilde{q})\mathbf{n} = 0 \ \text{ on } \Sigma^\infty_n, \end{cases}$$

admits a unique solution

$$\widetilde{\mathbf{z}} \in L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \ \text{ and } \ \widetilde{q} \in L^2(0,\infty;H^1_\delta(\Omega)). \tag{5.8.35}$$

Thus, since $\widetilde{\mathbf{v}}_2 = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$ and $\widetilde{p}_2 = \widetilde{\pi} + \widetilde{q}$, from (5.8.33), (5.8.34) and (5.8.35) we deduce that

$$\widetilde{\mathbf{v}}_2 \in L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)) \ \text{ and } \ \widetilde{p}_2 \in L^2(0,\infty;H^1_\delta(\Omega)).$$

This concludes the proof of (5.8.24).

As a consequence of (5.8.23) and (5.8.24), we obtain (5.8.21).

- **Step 5**. *Conclusion.*

Finally, using the fact that $\mathbf{v} = \widetilde{\mathbf{v}} + \mathbf{w} + \mathbf{z}$ and $p = \widetilde{p} + \pi$, together with (5.8.21) and Propositions 5.8.1 and 5.8.2, we obtain (5.8.12). This completes the proof. ☐

5.9 Stabilization of the nonlinear system

This section is divided into two parts. We first derive the estimates for the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ and $\widehat{F}_s$ in Subsection 5.9.1. Then, in Subsection 5.9.2, we prove Theorem 5.2.1.

Let us first establish an auxiliary result that will be used throughout Subsection 5.9.1. This result is a direct consequence of Definition 5.2.6.

Lemma 5.9.1. *There exists a positive constant C such that for all $\eta \in E(0,\infty)$, the extensions $\eta^{\pm}$ defined in (5.2.6) satisfy*

$$\begin{aligned}\|\partial_{z_1}\eta^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))} + \|\partial_t \eta^{\pm}\|_{L^\infty(0,\infty;H^{2+\alpha_0}(-L/2,L))}\\ + \|\partial^2_{z_1 t}\eta^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))} \le C\|\eta\|_{H^{4,2}((0,\infty)\times 0,L))}.\end{aligned} \tag{5.9.1}$$

5.9.1 Estimate of nonlinear terms

In this subsection, we derive the estimates for the nonlinear terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ and $\widehat{F}_s$.

We first recall that the space $\mathcal{Z}_\infty$ is defined

$$\begin{aligned}\mathcal{Z}_\infty = \Big(L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\alpha,2}_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))\Big)\\ \times L^2(0,\infty;H^{\frac{1}{2}+\alpha,1}_\delta(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s)),\end{aligned}$$

endowed with its natural norm.

Let $R > 0$, to be determined later (see (5.9.26)). Let us recall the definition of the set $B_\infty(R,\mathbf{u}_0,\eta_2^0)$:

$$\begin{aligned}B_\infty(R,\mathbf{u}_0,\eta_2^0) :=\Big\{(\widehat{\mathbf{u}},\widehat{p},\eta) \in \mathcal{Z}_\infty \mid \|(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\|_{\mathcal{Z}_\infty} \le R,\ \ \widehat{\eta} \in E(0,\infty)\\ \text{and } \widehat{\mathbf{u}}(0) = \mathbf{u}_0,\ \widehat{\eta}(0) = 0,\ \widehat{\eta}_t(0) = \eta_2^0\Big\}.\end{aligned} \tag{5.9.2}$$

Estimate of $\widehat{\mathbf{F}}_f$

Lemma 5.9.2. *There exists $C_{\widehat{\mathbf{F}}_f} > 0$ such that for all $(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have*

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} \le C_{\widehat{\mathbf{F}}_f}(1+R)R^3. \tag{5.9.3}$$

Furthermore, for all $(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1),(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have

$$\begin{aligned}\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1) - \widehat{\mathbf{F}}_f(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2)\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\\ \le C_{\widehat{\mathbf{F}}_f}(1+R)R^3\|(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1) - (\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2)\|_{\mathcal{Z}_\infty}.\end{aligned} \tag{5.9.4}$$

Proof. Due to the similarity between some of the terms present in $\widehat{\mathbf{F}}_f$, we will only show the estimates (5.9.3) and (5.9.4) for some of them. In particular, we selected the following terms:

- $\widetilde{\mathbf{F}}^1_f := \det(J)(\partial_{z_1}\widehat{\eta}^{\pm})(\widehat{\eta}^{\pm}_\chi)\dfrac{\partial \widehat{u}_i}{\partial z_1}\widehat{u}_1,$
- $\widetilde{\mathbf{F}}^2_f := \det(J)(\widehat{\eta}^{\pm}_t)(\widehat{\eta}^{\pm}_\chi)\dfrac{\partial \widehat{u}_i}{\partial z_2},$
- $\widetilde{\mathbf{F}}^3_f := (\det(J))^2(\partial_{z_1}\widehat{\eta}^{\pm})^2\dfrac{\partial^2 \widehat{u}_i}{\partial z_2 \partial z_1},$

- $\widetilde{\mathbf{F}}_f^4 := \det(J)(\partial_{z_1}\widehat{\eta}^{\pm})(\partial_{z_1}^2\widehat{\eta}^{\pm})\dfrac{\partial \widehat{u}_i}{\partial z_2}$,

where

$$\eta^+(t,\cdot,e) := (\mathcal{E}\eta)(t,\cdot) \text{ on } [-L/2,L], \quad \eta^-(t,\cdot,-e) := (\mathcal{E}\eta)(t,\cdot) \text{ on } [-L/2,L],$$

and $\det(J) = \frac{\ell-e}{\ell-e+\eta_\chi^{\pm}(t,z)}$, with $\eta_\chi^{\pm}(t,z) := (\mp\chi^{\pm} + (\ell \mp z_2)\partial_{z_2}\chi^{\pm})\,\widehat{\eta}^{\pm}(t,z_1)$, $(t,z) \in [0,\infty)\times\overline{\Omega}$. Secondly, let us observe that in order to show the estimate (5.9.3), it is sufficient to prove

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega))} \le C_{\widehat{\mathbf{F}}_f}(1+R)^2R^4 \tag{5.9.5}$$

and

$$\|\widehat{\mathbf{F}}_f(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})|_{\Omega_2}\|_{L^2(0,\infty;\mathbf{L}^2(\Omega_2))} \le C_{\widehat{\mathbf{F}}_f}(1+R)^2R^4. \tag{5.9.6}$$

The Lipschitz estimate (5.9.4) can be obtained in a similar way to how we will proceed to prove (5.9.3).

- $\widetilde{\mathbf{F}}_f^1$: Using Hölder's inequality, Lemma 5.9.1, the continuous embedding $H^{\frac{1}{2}+\alpha}(\Omega) \hookrightarrow L^4(\Omega)$ and [MRR20, Lemma A.4], we obtain

$$\begin{aligned}
&\left\|\det(J)(\partial_{z_1}\widehat{\eta}^{\pm})\frac{\partial \widehat{u}_i}{\partial z_1}\widehat{u}_1\right\|_{L^2(L^2(\Omega))} \\
&\le C\,\|\det(J)\|_{L^\infty((0,\infty)\times\Omega)}\,\|\partial_{z_1}\widehat{\eta}^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))}\|\widehat{\eta}^{\pm}\|_{L^\infty((0,T)\times(-L/2,L))} \\
&\qquad\qquad \times \|\widehat{u}_1\|_{L^\infty(H^{\frac{1}{2}+\alpha}(\Omega))} \times \left\|\frac{\partial \widehat{u}_i}{\partial z_1}\right\|_{L^2(H^{\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\left(1+\|\widehat{\eta}^{\pm}\|_{L^\infty(H^2(-L/2,L))}\right)\|\widehat{\eta}^{\pm}\|_{L^\infty(H^2(-L/2,L))}\|\partial_{z_1}\widehat{\eta}^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))} \\
&\qquad \times \|\widehat{\eta}^{\pm}\|_{L^\infty((0,T)\times(-L/2,L))}\|\widehat{u}_1\|_{L^\infty(H^{\frac{1}{2}+\alpha}(\Omega))} \times \left\|\frac{\partial \widehat{u}_i}{\partial z_1}\right\|_{L^2(H^{\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\,(1+R)\,R^5.
\end{aligned}$$

- $\widetilde{\mathbf{F}}_f^2$: After using Lemma 5.9.1 and [MRR20, Lemma A.4], we obtain

$$\begin{aligned}
&\left\|\det(J)(\widehat{\eta}_t^{\pm})(\widehat{\eta}_\chi^{\pm})\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(L^2(\Omega))} \\
&\le C\,\|\det(J)\|_{L^\infty((0,\infty)\times\Omega)}\,\|\eta_t^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))}\|\widehat{\eta}^{\pm}\|_{L^\infty((0,T)\times(-L/2,L))}\left\|\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(L^2(\Omega))} \\
&\le C\left(1+\|\widehat{\eta}^{\pm}\|_{L^\infty(H^2(-L/2,L))}\right)\|\widehat{\eta}^{\pm}\|_{L^\infty(H^2(-L/2,L))} \\
&\qquad \times \|\eta_t^{\pm}\|_{L^\infty((0,\infty)\times(-L/2,L))}\|\widehat{\eta}^{\pm}\|_{L^\infty((0,T)\times(-L/2,L))}\left\|\frac{\partial \widehat{u}_i}{\partial z_2}\right\|_{L^2(L^2(\Omega))} \\
&\le C\,(1+R)\,R^4.
\end{aligned}$$

- $\widetilde{\mathbf{F}}_f^3$: From Lemma 5.9.1 and [MRR20, Lemma A.4], we get

$$\begin{aligned}
&\left\|(\det(J))^2(\partial_{z_1}\widehat{\eta}^{\pm})^2\frac{\partial^2 \widehat{u}_i}{\partial z_2\partial z_1}\right\|_{L^2(H^{-\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\,\|\det(J)\|^2_{L^\infty((0,\infty)\times\Omega)}\,\|\partial_{z_1}\widehat{\eta}^{\pm}\|^2_{L^\infty((0,\infty)\times(-L/2,L))}\left\|\frac{\partial^2 \widehat{u}_i}{\partial z_2\partial z_1}\right\|_{L^2(H^{-\frac{1}{2}+\alpha}(\Omega))} \\
&\le C\,(1+R)^2\,R^5.
\end{aligned}$$

• $\widetilde{\mathbf{F}}_f^4$: After using Hölder's inequality together with the continuous injections $H^{\frac12+\alpha}(\Omega)\hookrightarrow L^4(\Omega)$ and $H^{a_0}(-L/2,L)\hookrightarrow L^4(-L/2,L)$ (with $a_0\in(1/4,1/2)$ being the parameter that appears in Proposition 5.2.2), and Lemma 5.9.1 and [MRR20, Lemma A.4], we obtain

$$\begin{aligned}&\left\|\det(J)(\partial_{z_1}\widehat{\eta}^\pm)(\partial_{z_1}^2\widehat{\eta}^\pm)\frac{\partial\widehat{u}_i}{\partial z_2}\right\|_{L^2(L^2(\Omega))}\\&\le C\,\|\det(J)\|_{L^\infty((0,\infty)\times\Omega)}\,\|\partial_{z_1}\widehat{\eta}^\pm\|_{L^\infty((0,\infty)\times(-L/2,L))}\|\partial_{z_1}^2\widehat{\eta}^\pm\|_{L^\infty(H^{a_0}(-L/2,L))}\left\|\frac{\partial\widehat{u}_i}{\partial z_2}\right\|_{L^2(L^2(\Omega))}\\&\le C\,(1+R)\,R^4.\end{aligned}$$

This completes the proof. □

Estimate of $\widehat{\mathbf{G}}_{\rm div}$

Lemma 5.9.3. *There exists $C_{\widehat{\mathbf{G}}_{\rm div}}>0$ such that for all $(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have*

$$\left\|{\rm div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}\right)(\widehat{\mathbf{u}},\widehat{\eta})\right\|_{L^2(0,\infty;\mathbf{H}^{\frac12+\alpha,1}(\Omega))}+\|\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}},\widehat{\eta})\|_{H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega))}\le C_{\widehat{\mathbf{G}}_{\rm div}}(1+R)R^2 \tag{5.9.7}$$

and

$$\left\|(1-\widetilde{\theta})\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}},\widehat{\eta})\right\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac32+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega))}\le C_{\widehat{\mathbf{G}}_{\rm div}}(1+R)R^2, \tag{5.9.8}$$

where $\widetilde{\theta}$ is introduced in (3.4.7). Furthermore, for all $(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1)$ and $(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2)\in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have

$$\begin{aligned}&\left\|\left({\rm div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}\right)(\widehat{\mathbf{u}}^1,\widehat{\eta}^1)-{\rm div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}\right)(\widehat{\mathbf{u}}^2,\widehat{\eta}^2)\right)\right\|_{L^2(0,\infty;\mathbf{H}^{\frac12+\alpha,1}(\Omega))}\\&+\left\|\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}}^1,\widehat{\eta}^1)-\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}}^2,\widehat{\eta}^2)\right\|_{H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega))}\\&+\left\|(1-\widetilde{\theta})\left(\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}}^1,\widehat{\eta}^1)-\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}}^2,\widehat{\eta}^2)\right)\right\|_{L^2(0,\infty;\mathbf{H}_{2,\delta}^{\frac32+\alpha}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac12+\alpha,0}(\Omega))}\\&\le C_{\widehat{\mathbf{G}}_{\rm div}}(1+R)R^2\|(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1)-(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2)\|_{\mathcal{Z}_\infty}.\end{aligned} \tag{5.9.9}$$

Proof. Let us recall that $\widehat{\mathbf{G}}_{\rm div}$ is given by

$$\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}},\widehat{\eta})=-\frac{\eta_\chi^\pm}{\ell-e}\widehat{u}_1\vec{\mathbf{e}}_1+\frac{(\ell\mp z)\chi^\pm\partial_{z_1}\widehat{\eta}^\pm}{\ell-e}\widehat{u}_1\vec{\mathbf{e}}_2 \tag{5.9.10}$$

and

$${\rm div}\,\widehat{\mathbf{G}}_{\rm div}(\widehat{\mathbf{u}},\widehat{\eta})=-\frac{\eta_\chi^\pm}{\ell-e}\frac{\partial\widehat{u}_1}{\partial z_1}+\frac{(\ell\mp z)\chi^\pm\partial_{z_1}\widehat{\eta}^\pm}{\ell-e}\frac{\partial\widehat{u}_1}{\partial z_2}. \tag{5.9.11}$$

We will only present the proof of the estimates (5.9.7) and (5.9.8). The proof of the Lipschitz estimate (5.9.9) can be proved in a similar way. The proof of estimates (5.9.7) and (5.9.8) is divided into three steps.

• Step 1: *Proof of estimate* $\left\|{\rm div}\left(\widetilde{\theta}\widehat{\mathbf{G}}_{\rm div}\right)(\widehat{\mathbf{u}},\widehat{\eta})\right\|_{L^2(\mathbf{H}^{\frac12+\alpha,1})}\le CR^2$.

We will only consider the term $(\partial_{z_1}\widehat{\eta}^\pm)\frac{\partial\widehat{u}_1}{\partial z_2}$. To estimate the remaining terms we can proceed in a similar way. Let us notice that it suffices to show that

$$\left\|(\partial_{z_1}\widehat{\eta}^\pm)\frac{\partial\widehat{u}_1}{\partial z_2}\right\|_{L^2(L^2)}+\left\|\nabla\left((\partial_{z_1}\widehat{\eta}^\pm)\frac{\partial\widehat{u}_1}{\partial z_2}\right)\right\|_{L^2(\mathbf{H}^{-\frac12+\alpha,0})}\le CR^2. \tag{5.9.12}$$

The proof of (5.9.12) is similar to the one presented in the proof of Lemma 5.9.2.

• Step 2: *Proof of estimate* $\|\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{u}},\widehat{\eta})\|_{H^1(\mathbf{H}^{-\frac{1}{2}+\alpha,0})} \leq CR^2$.

It suffices to show that

$$\|(\partial_{z_1}\widehat{\eta}^{\pm})\widehat{u}_1\|_{L^2(H^{-\frac{1}{2}+\alpha})} \leq CR^2 \tag{5.9.13}$$

and

$$\|\partial_t((\partial_{z_1}\widehat{\eta}^{\pm})\widehat{u}_1)\|_{L^2(H^{-\frac{1}{2}+\alpha})} \leq CR^2. \tag{5.9.14}$$

The arguments used to prove estimates (5.9.13) and (5.9.14) are similar to those presented in the proof of Lemma 5.9.3.

Thus, from steps 1 and 2, we deduce estimate (5.9.7).

• Step 3: *Proof of estimate* (5.9.8).
From Lemma 5.9.1, we obtain

$$\begin{aligned}\|(1-\widetilde{\theta})(\partial_{z_1}\widehat{\eta}^{\pm})\widehat{u}_1\|_{L^2(H^{\frac{3}{2}+\alpha})} &\leq C\|\partial_{z_1}\widehat{\eta}^{\pm}\|_{L^\infty((0,\infty)\times(\varepsilon_0,L))}\|\widehat{u}_1\|_{L^2(H^{\frac{3}{2}+\alpha})}\\ &\leq CR^2.\end{aligned} \tag{5.9.15}$$

The estimate of $(1-\widetilde{\theta})(\partial_{z_1}\widehat{\eta}^{\pm})\widehat{u}_1$ in the $H^1(H^{-\frac{1}{2}+\alpha})$−norm can be done using the same argument used in step 2. This completes the proof estimate (5.9.8). □

Estimate of $\widehat{F}_s$.

Lemma 5.9.4. *There exist $C_{\widehat{F}_s} > 0$ such that for all $(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have*

$$\|\widehat{F}_s(\widehat{\mathbf{u}},\widehat{\eta})\|_{L^2(0,\infty;H^\alpha(0,\ell_s))} \leq C_{\widehat{F}_s}(1+R)R^4. \tag{5.9.16}$$

Furthermore, for all $(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1),(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have

$$\begin{aligned}&\|\widehat{F}_s(\widehat{\mathbf{u}}^1,\widehat{\eta}^1)-\widehat{F}_s(\widehat{\mathbf{u}}^2,\widehat{\eta}^2)\|_{L^2(0,\infty;H^\alpha(0,\ell_s))}\\ &\qquad\leq C_{\widehat{F}_s}(1+R)R^4\|(\widehat{\mathbf{u}}^1,\widehat{p}^1,\widehat{\eta}^1)-(\widehat{\mathbf{u}}^2,\widehat{p}^2,\widehat{\eta}^2)\|_{\mathcal{Z}_\infty}.\end{aligned} \tag{5.9.17}$$

Proof. We will only consider the term

$$\gamma_s^+\left(\det(J)(\partial_{z_1}\widehat{\eta}^{\pm})^2(\widehat{\eta}_\chi^{\pm})\frac{\partial\widehat{u}_{s,2}}{\partial z_2}\right).$$

The Lipschitz estimate (5.9.17) can be proved in a similar way as estimate (5.9.16). Let us start by proving the following estimate:

$$\left\|\det(J)(\partial_{z_1}\widehat{\eta}^{\pm})^2(\widehat{\eta}_\chi^{\pm})\frac{\partial\widehat{u}_{s,2}}{\partial z_2}\right\|_{L^2(H^{\frac{1}{2}+\alpha})} \leq C(1+R)R^4. \tag{5.9.18}$$

To prove the estimate (5.9.18), we will show that

$$\left\|\det(J)(\partial_{z_1}\widehat{\eta}^{\pm})^2(\widehat{\eta}_\chi^{\pm})\frac{\partial\widehat{u}_{s,2}}{\partial z_2}\right\|_{L^2(L^2)} \leq C(1+R)R^4 \tag{5.9.19}$$

and

$$\left\|\nabla\left(\det(J)(\partial_{z_1}\widehat{\eta}^{\pm})^2(\widehat{\eta}_\chi^{\pm})\frac{\partial\widehat{u}_{s,2}}{\partial z_2}\right)\right\|_{L^2(\mathbf{H}^{-\frac{1}{2}+\alpha})} \leq C(1+R)R^4 \tag{5.9.20}$$

Estimates (5.9.19) and (5.9.20) follow by arguments similar to those used in the proof of Lemma 5.9.2. This completes the proof. □

5.9.2 Proof of Theorem 5.2.1

Let us first recall the definition of the space $\mathcal{Z}_\infty$:

$$\mathcal{Z} = \Big(L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega)) \cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))\Big) \times L^2(0,\infty;H_\delta^{\frac{1}{2}+\alpha,1}(\Omega)) \times H^{4,2}_{\{0,\ell_s\}}((0,\infty)\times(0,\ell_s)),$$

We now introduce the mapping $\mathcal{N}$ from $\mathcal{Z}_\infty$ into itself, defined by

$$\mathcal{N}(\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k}) = (\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}) \quad \text{for all } (\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k}) \in \mathcal{Z}_\infty, \tag{5.9.21}$$

where $(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})$ is the solution of the system

$$\begin{cases}\partial_t\widehat{\mathbf{u}} - \operatorname{div}\sigma(\widehat{\mathbf{u}},\widehat{p}) - A_1\widehat{\eta} - A_2\widehat{\eta}_t = e^{-\omega t}\widehat{\mathbf{F}}_f(\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k}) \ \text{ in } Q^\infty,\\ \operatorname{div}\widehat{\mathbf{u}} = A_3\widehat{\eta}_1 + e^{-\omega t}\operatorname{div}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{\Phi}},\widehat{k}) \ \text{ in } Q^\infty,\\ \widehat{\mathbf{u}} = \widehat{\mathbf{g}}_p \ \text{ on } \Sigma_i^\infty, \ \ \widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_w^\infty\cup\Sigma_r^\infty, \ \ \widehat{\mathbf{u}} = \widehat{\eta}_t\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_s^\infty,\\ \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty, \ \ \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 \ \text{ in } \Omega,\\ \partial_t^2\widehat{\eta} + \alpha\Delta_s^2\widehat{\eta} + \gamma(\Delta_s^2)^{\frac{1}{2}}\widehat{\eta}_t = -\gamma_s^{+,-}\widehat{p} + e^{-\omega t}\widehat{F}_s(\widehat{\mathbf{\Phi}},\widehat{k}) + \sum_{i=1}^{N_c}\mathcal{K}_i(\widehat{\mathbf{u}},\widehat{\eta},\widehat{\eta}_t)w_i \ \text{ in } (0,\infty)\times(0,\ell_s),\\ \widehat{\eta} = 0 \ \text{ and } \ \partial_{x_1}\widehat{\eta} = 0 \ \text{ on } (0,\infty)\times\{0\},\\ \partial_{x_1}^2\widehat{\eta} = 0 \ \text{ and } \ \partial_{x_1}^3\widehat{\eta} = 0 \ \text{ on } (0,\infty)\times\{\ell_s\},\\ \widehat{\eta}(0) = 0 \ \text{ and } \ \widehat{\eta}_t(0) = \eta_2^0 \ \text{ in } (0,\ell_s).\end{cases} \tag{5.9.22}$$

Proposition 5.9.1. *There exists a constant $C>0$ such that for all $\mathbf{u}^0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H^2_{\{0\}}(0,\ell_s)$ and $\widehat{\mathbf{g}}_p \in H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))$, the mapping $\mathcal{N}$ is well-defined. Moreover, for all $(\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k}) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$ and all $(\widehat{\mathbf{\Phi}}^1,\widehat{\psi}^1,\widehat{k}^1)$, $(\widehat{\mathbf{\Phi}}^2,\widehat{\psi}^2,\widehat{k}^2) \in B_\infty(R,\mathbf{u}_0,\eta_2^0)$, we have*

$$\|\mathcal{N}(\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k})\|_{\mathcal{Z}_\infty} \le C\Big(\|\widehat{\mathbf{u}}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\widehat{\mathbf{g}}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))} + p(R)\Big) \tag{5.9.23}$$

and

$$\|\mathcal{N}(\widehat{\mathbf{\Phi}}^1,\widehat{\psi}^1,\widehat{k}^1) - \mathcal{N}(\widehat{\mathbf{\Phi}}^2,\widehat{\psi}^2,\widehat{k}^2)\|_{\mathcal{Z}_\infty} \le Cp(R)\|(\widehat{\mathbf{\Phi}}^1,\widehat{\psi}^1,\widehat{k}^1) - (\widehat{\mathbf{\Phi}}^2,\widehat{\psi}^2,\widehat{k}^2)\|_{\mathcal{Z}_\infty}, \tag{5.9.24}$$

where $p = p(R)$ is a polynomial of degree greater than 1 such that $p(0) = 0$.

Proof. From Theorem 5.8.1 and Proposition 5.8.2, we obtain

$$\begin{aligned}\|(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta})\|_{\mathcal{Z}_\infty} \le C_L\Big(&\|\widehat{\mathbf{u}}_0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\widehat{\mathbf{g}}_p\|_{H^1_{\{0\}}(0,\infty;\mathbf{H}(\Gamma_i))}\\ &+ \|\widehat{\mathbf{F}}_f(\widehat{\mathbf{\Phi}},\widehat{\psi},\widehat{k})\|_{L^2(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))} + \|\widehat{F}_s(\widehat{\mathbf{\Phi}},\widehat{k})\|_{L^2((0,\infty)\times(0,\ell_s))}\\ &+ \Big\|\operatorname{div}\Big(\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{\Phi}},\widehat{k})\Big)\Big\|_{L^2(0,\infty;\mathbf{H}^{\frac{1}{2}+\alpha,1}(\Omega))} + \|\widetilde{\theta}\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{\Phi}},\widehat{k})\|_{H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\\ &+ \Big\|(1-\widetilde{\theta})\widehat{\mathbf{G}}_{\mathrm{div}}(\widehat{\mathbf{\Phi}},\widehat{k})\Big\|_{L^2(0,\infty;\mathbf{H}_\delta^{\frac{3}{2}+\alpha,2}(\Omega))\cap H^1(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha,0}(\Omega))}\Big),\end{aligned} \tag{5.9.25}$$

where $C_L > 0$ is the constant appearing in estimate (5.8.12). After using Lemmas 5.9.2, 5.9.3

and 5.9.4, we obtain

$$\|(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta})\|_{\mathcal{Z}_\infty} \leq C_L \left(\|\widehat{\mathbf{u}}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(0,\ell_s)} + \|\widehat{\mathbf{g}}_p\|_{H^1(0,\infty;\mathbf{H}(\Gamma_i))} + p(R) \right).$$

The proof of the Lipschitz estimate (5.9.24) can be established by a similar argument. □

We are now in position to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. We first choose $R > 0$ small enough such that

$$Cp(R) < \min\left\{\frac{R}{2}, 1\right\} \quad \text{and} \quad CR < \frac{\ell - e}{2}, \tag{5.9.26}$$

where $p = p(R)$ is the polynomial of degree greater than 1 appearing in Proposition 5.9.1 satisfying $p(0) = 0$. Then, by choosing $r > 0$ such that $C_L r < \frac{R}{2}$, Proposition 5.9.1 allows us to deduce that the operator $\mathcal{N}$, defined in (5.9.21), maps $B_\infty(R, \mathbf{u}_0, \eta_2^0)$ into itself and is a strict contraction. Finally, thanks to the Banach fixed point Theorem, system (5.9.22) admits a unique solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}) \in B_\infty(R, \mathbf{u}_0, \eta_2^0)$. This completes the proof of Theorem 5.2.1.

Appendix A: Nonlinear terms coming from the geometric transformation (5.2.8)

In this section, we present the explicit formulas of the terms $\widehat{\mathbf{F}}_f$, $\widehat{\mathbf{G}}_{\text{div}}$ and $\widehat{F}_s$ that we obtain for the geometric transformation given in (5.2.8).

$$
\begin{aligned}
&\widehat{\mathbf{F}}_{f,i}(\widehat{\mathbf{u}},\widehat{p},\widehat{\eta}_1,\widehat{\eta}_2) = -e^{-\omega t}\left(\widehat{u}_1\widehat{u}_{i,z_1}+\widehat{u}_2\widehat{u}_{i,z_2}\right)+\frac{e^{-2\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_2\widehat{u}_{i,z_2}+\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_2 u_{s,i,z_2}\\
&+\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_{i,z_2}u_{s,2}+\frac{e^{-\omega t}(\widehat{\eta}_\chi^\pm)^2}{\ell-e}\left(e^{-2\omega t}\widehat{u}_2\widehat{u}_{i,z_2}+e^{-\omega t}\widehat{u}_2 u_{s,i,z_2}+e^{-\omega t}u_{s,2}\widehat{u}_{i,z_2}+u_{s,2}u_{s,i,z_2}\right)\\
&+\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_\chi^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{u}_1+u_{s,1}\right)\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)+\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_2^\pm}{\ell-e}\widehat{u}_{i,z_2}\\
&-\frac{e^{-2\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_2^\pm\widehat{\eta}_\chi^\pm}{\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm}\widehat{u}_{i,z_2}-2\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_{i,z_1z_2}+2\nu\frac{e^{-2\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_\chi^\pm}{\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm}\widehat{u}_{i,z_1z_2}\\
&-\frac{e^{-\omega t}(\ell\mp z_2)\widehat{\eta}_2^\pm\widehat{\eta}_\chi^\pm}{\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm}u_{s,i,z_2}+2\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_\chi^\pm}{\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm}u_{s,i,z_1z_2}\\
&+\nu\frac{e^{-\omega t}\left((\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\right)^2}{(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)^2}\left(e^{-\omega t}\widehat{u}_{i,z_1z_2}+u_{s,i,z_1z_2}\right)-2\nu\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_{i,z_2z_2}\\
&-\nu\frac{e^{-\omega t}(\widehat{\eta}_\chi^\pm)^2}{(\ell-e)^2}\left(e^{-\omega t}\widehat{u}_{i,z_2z_2}+u_{s,z_2z_2}\right)+2\nu\frac{e^{-\omega t}(2(\ell-e)+e^{-\omega t}\widehat{\eta}_\chi^\pm)(\widehat{\eta}_\chi^\pm)^2}{(\ell-e)^3}\left(e^{-\omega t}\widehat{u}_{i,z_2z_2}+u_{s,i,z_2z_2}\right)\\
&+2\nu\frac{e^{-2\omega t}(2(\ell-e)+e^{-\omega t}\widehat{\eta}_\chi^\pm)(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)^3(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{u}_{i,z_2z_2}+u_{s,i,z_2z_2}\right)-\nu\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_{i,z_2}\\
&-\nu\frac{e^{-2\omega t}(2(\ell-e)+e^{-\omega t}\widehat{\eta}_\chi^\pm)(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)^2(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)^2}\left(e^{-\omega t}\widehat{u}_{i,z_2z_2}+u_{s,i,z_2z_2}\right)-\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1z_1}^\pm}{\ell-e}u_{s,i,z_2}\\
&+\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_{1,z_1z_1}^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&-\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_{\chi,z_1}^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&+\nu\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_{\chi,z_1}^\pm}{\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm}\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&-\nu\frac{e^{-2\omega t}(\ell\mp z_2)^2(\chi\widehat{\eta}_{1,z_1}^\pm)^2\widehat{\eta}_{\chi,z_2}^\pm}{(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)^3}\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&+\nu\left(\frac{2e^{-\omega t}(\widehat{\eta}_\chi^\pm)^2}{(\ell-e)^2}-\frac{2e^{-\omega t}(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)^2(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}-\frac{e^{-2\omega t}(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)^2}\right)\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&+\nu\frac{\widehat{\eta}_\chi^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{(\ell-e)^2}-\frac{2e^{-\omega t}(\widehat{\eta}_\chi^\pm)^2}{(\ell-e)^3}\right)\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&+\nu\frac{\widehat{\eta}_\chi^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(\frac{2e^{-\omega t}(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)^3(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}+\frac{e^{-3\omega t}(\widehat{\eta}_\chi^\pm)^3}{(\ell-e)^2(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)^2}\right)\left(e^{-\omega t}\widehat{u}_{i,z_2}+u_{s,i,z_2}\right)\\
&\nu\left(\frac{e^{-\omega t}\widehat{\eta}_{\chi,z_i}^\pm}{\ell-e}\widehat{u}_{1,z_1}+\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{u}_{1,z_1z_i}\right)-\nu e^{-\omega t}\left(\partial_{z_i}\left(\frac{(\ell\mp z_2)\chi^\pm}{\ell-e}\right)\widehat{\eta}_{1,z_1}^\pm-\left(\frac{(\ell\mp z_2)\chi^\pm}{\ell-e}\right)\widehat{\eta}_{1,z_1z_i}^\pm\right)\widehat{u}_{1,z_2}\\
&-\nu e^{-\omega t}\left(\frac{(\ell\mp z_2)\chi^\pm}{\ell-e}\right)\widehat{\eta}_{1,z_1}^\pm\widehat{u}_{1,z_1z_i}+\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm}{\ell-e}\widehat{p}_{z_2}\delta_{1,i}\\
&-\frac{e^{-\omega t}(\ell\mp z_2)\chi^\pm\widehat{\eta}_{1,z_1}^\pm\widehat{\eta}_\chi^\pm}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{p}_{z_2}+p_{s,z_2}\right)\delta_{1,i}+\frac{e^{-\omega t}\widehat{\eta}_\chi^\pm}{\ell-e}\widehat{p}_{z_2}\delta_{2,i}\\
&-\frac{e^{-\omega t}(\widehat{\eta}_\chi^\pm)^2}{(\ell-e)(\ell-e+e^{-\omega t}\widehat{\eta}_\chi^\pm)}\left(e^{-\omega t}\widehat{p}_{z_2}+p_{s,z_2}\right)\delta_{i,2}
\end{aligned}
\tag{5.9.27}
$$

and

$$
\begin{aligned}
&\mathbf{G}_{\mathrm{div}}(\widehat{\mathbf{u}},\widehat{\eta}_1) = -\frac{e^{-\omega t}\widehat{\eta}_\chi^{\pm}}{\ell - e}\widehat{u}_1\vec{\mathbf{e}}_1 + \frac{e^{-\omega t}(\ell \mp z_2)\chi^{\pm}\widehat{\eta}_{1,z_1}^{\pm}}{\ell - e}\widehat{u}_1\vec{\mathbf{e}}_2,\\
&\widehat{F}_s(\widehat{\mathbf{u}},\widehat{\eta}_1) = -\nu e^{-\omega t}\widehat{\eta}_{1,z_1}^{\pm}\gamma_s^{\pm}\widehat{u}_{1,z_2} + \nu\frac{e^{-2\omega t}\widehat{\eta}_{1,z_1}^{\pm}\widehat{\eta}_\chi^{\pm}\gamma_s^{\pm}\widehat{u}_{1,z_2}}{\ell - e + e^{-\omega t}\widehat{\eta}_\chi^{\pm}} - \nu e^{-\omega t}\widehat{\eta}_{1,z_1}^{\pm}\gamma_s^{\pm}\widehat{u}_{2,z_1}\\
&+\nu\frac{e^{-2\omega t}(\ell \mp z_2)(\widehat{\eta}_{1,z_1}^{\pm})^2}{\ell - e}\gamma_s^{\pm}\widehat{u}_{2,z_2} - \nu\frac{e^{-2\omega t}(\ell \mp z_2)(\widehat{\eta}_{1,z_1}^{\pm})^2\widehat{\eta}_\chi^{\pm}}{(\ell - e)(\ell - e + e^{-\omega t}\widehat{\eta}_\chi^{\pm})}\gamma_s^{\pm}u_{s,2,z_2}\\
&+\nu\frac{e^{-\omega t}\widehat{\eta}_{1,z_1}^{\pm}\widehat{\eta}_\chi^{\pm}}{\ell - e + e^{-\omega t}\widehat{\eta}_\chi^{\pm}}\gamma_s^{\pm}u_{s,1,z_2} + \nu\frac{e^{-\omega t}(\ell \mp z_2)(\widehat{\eta}_{1,z_1}^{\pm})^2}{\ell - e}\gamma_s^{\pm}u_{s,2,z_2}\\
&-\nu\frac{e^{-2\omega t}(\ell \mp z_2)(\widehat{\eta}_{1,z_1}^{\pm})^2\widehat{\eta}_\chi^{\pm}}{(\ell - e)(\ell - e + e^{-\omega t}\widehat{\eta}_\chi^{\pm})}\gamma_s^{\pm}u_{s,2,z_2} + 2\nu\frac{\ell - e}{\ell - e + e^{-\omega t}\widehat{\eta}_\chi^{\pm}}\gamma_s^{\pm}\widehat{u}_{2,z_2},
\end{aligned}
\tag{5.9.28}
$$

where $\delta_{i,j}$ denotes the Kronecker delta.

Appendix B: Proof of Theorem 5.3.2

Let us first notice notice that the domain $\mathcal{D}(A; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$ of the Oseen operator is dense in $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$, since $\mathcal{D}(A; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)) = \mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$ and $\mathcal{D}(A_0; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega))$ is dense in $\mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)$ (see Proposition 2.4.8 of Chapter 2). Then, thanks to [EN06, Theorem 4.6, p. 95], it suffices to verify that $(A, \mathcal{D}(A; \mathbf{V}_{n,\Gamma_d}^{-\frac{1}{2}+\alpha}(\Omega)))$ is sectorial.

We split the proof into three steps.

Step 1: (*Estimate in* $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$−norm). Since $(A_0, \mathcal{D}(A; \mathbf{V}^0_{n,\Gamma_d}(\Omega)))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ (see [NR15, Theorem 2.8]), there exist $C_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ such that

$$\|(\lambda I - A)^{-1}\mathbf{F}\|_{\mathbf{L}^2} \leq \frac{C_0}{|\lambda|}\|\mathbf{F}\|_{\mathbf{L}^2}, \tag{5.9.29}$$

for all $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$ and all $\mathbf{F} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$.

Step 2: (*Estimate in* $\mathbf{V}^{-1}_{\Gamma_d}(\Omega)$−norm). We claim that there exists $C_1 > 0$ such that

$$\|(\lambda I - A)^{-1}\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}} \leq \frac{C_1}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}} \tag{5.9.30}$$

for all $\lambda \in \Sigma_{\theta_0} \setminus \{0\}$ and all $\mathbf{F} \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Indeed, let $\mathbf{u} := (\lambda I - A)^{-1}\mathbf{F}$. Then,

$$\begin{cases} \lambda\mathbf{u} - \operatorname{div}\sigma(\mathbf{u}, q) + (\mathbf{u}\cdot\nabla)\mathbf{u}_s + (\mathbf{u}_s\cdot\nabla)\mathbf{u} = \mathbf{F} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \Gamma_d, \ \ \sigma(\mathbf{u}, q)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{5.9.31}$$

Thus, $\mathbf{u}$ satisfies in particular

$$\lambda \int_\Omega \mathbf{u}\cdot\overline{\boldsymbol{\varphi}} = -2\nu\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\overline{\boldsymbol{\varphi}}) - \int_\Omega ((\mathbf{u}_s\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_s)\cdot\overline{\boldsymbol{\varphi}} + \langle \mathbf{F}, \overline{\boldsymbol{\varphi}}\rangle_{\mathbf{V}^{-1}_{\Gamma_d}, \mathbf{V}^1_{\Gamma_d}}, \tag{5.9.32}$$

for all $\overline{\boldsymbol{\varphi}} \in \mathbf{V}^1_{\Gamma_d}(\Omega)$.

After properly adapting the argument of the proof of Theorem 2.4.4 presented at the end of Chapter 2 (possibly after modifiying the sector Σ_{θ_0}), we can show that there exists a constant $C > 0$, independent of λ, such that

$$\|\mathbf{u}\|_{\mathbf{V}^1_{\Gamma_d}} \leq C\|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}}. \tag{5.9.33}$$

We now introduce the set

$$\mathcal{D} := \left\{\boldsymbol{\varphi} \in \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \|\boldsymbol{\varphi}\|_{\mathbf{V}^1_{\Gamma_d}} \leq 1\right\}.$$

Using the Cauchy-Schwarz inequality, estimate (5.9.33), Assumption A1 and [BH21, Theorem 7.4] in identity (5.9.32), we obtain

$$\begin{aligned}
|\lambda|\|\mathbf{u}\|_{\mathbf{V}^{-1}_{\Gamma_d}} &= |\lambda| \sup_{\varphi\in\mathcal{D}} \left|\langle \mathbf{u}, \overline{\boldsymbol{\varphi}}\rangle_{\mathbf{V}^{-1}_{\Gamma_d},\mathbf{V}^{1}_{\Gamma_d}}\right| \\
&\leq 2\nu \sup_{\varphi\in\mathcal{D}} \int_\Omega |\varepsilon(\mathbf{u})||\varepsilon(\overline{\boldsymbol{\varphi}})| + \sup_{\varphi\in\mathcal{D}} \int_\Omega |((\mathbf{u}_s\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_s)|\,|\overline{\boldsymbol{\varphi}}| + \sup_{\varphi\in\mathcal{D}} \left|\langle \mathbf{F}, \overline{\boldsymbol{\varphi}}\rangle_{\mathbf{V}^{-1}_{\Gamma_d},\mathbf{V}^{1}_{\Gamma_d}}\right| \\
&\leq C\|\mathbf{F}\|_{\mathbf{V}^{-1}_{\Gamma_d}},
\end{aligned} \tag{5.9.34}$$

from where we deduce the estimate (5.9.30) with a constant $C > 0$ independent of λ.

Step 3. (Conclusion). From Lemma 3.3.1 of Chapter 3 and the interpolation between (5.9.29) and (5.9.30) we deduce that for any $\epsilon \in (0, 1/2)$

$$\|(\lambda I - A)^{-1}\mathbf{F}\|_{\mathbf{V}^{-\epsilon}_{\Gamma_d}} \leq C_0 \left(\frac{C_1}{C_0}\right)^{\epsilon} \frac{1}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}^{-\epsilon}_{\Gamma_d}} \text{ for all } \lambda \in \Sigma_{\theta_0}\setminus\{0\}. \tag{5.9.35}$$

Now, taking $\epsilon = 1/2 - \alpha \in (0, 1/2)$ in (5.9.35) and using the fact that $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is dense in $\mathbf{V}^{-\frac{1}{2}+\alpha}_{n,\Gamma_d}(\Omega)$ we deduce that

$$\|(\lambda I - A)^{-1}\mathbf{F}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} \leq \frac{C}{|\lambda|}\|\mathbf{F}\|_{\mathbf{V}^{-\frac{1}{2}+\alpha}_{\Gamma_d}} \text{ for all } \lambda \in \Sigma_{\theta_0}\setminus\{0\}. \tag{5.9.36}$$

Appendix C: Justification of the identity (5.4.14) (formal computations)

The aim of this appendix is to present the formal computations that lead to identity (5.4.14).

Let $\mathbf{F}_f, \mathbf{G}_f \in \mathbf{L}^2(\Omega)$, $F_s^1, G_s^1 \in H^2_{\{0\}}(0,\ell_s)$ and $F_s^2, G_s^2 \in L^2(0,\ell_s)$. Let us consider the following two systems:

$$\begin{cases} \lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},q) + (\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2 = \mathbf{F}_f \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v} = A_3\eta_1 \ \text{ in } \Omega, \\ \mathbf{v} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_d\setminus\Gamma_s, \quad \sigma(\mathbf{v},q)\mathbf{n} = 0 \ \text{ on } \Gamma_n, \\ \lambda\eta_1 - \eta_2 = F_s^1 \ \text{ on } (0,\ell_s), \\ \lambda\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\eta_2 - A_4\eta_1 = -\gamma_s^+ q + \gamma_s^- q + F_s^2 \ \text{ on } (0,\ell_s), \\ \eta_1(0) = \partial_x\eta(0) = 0 \ \text{ and } \ \partial_x^2\eta_1(\ell_s) = \partial_x^3\eta_1(\ell_s) = 0, \end{cases} \tag{5.9.37}$$

and

$$\begin{cases} \lambda\mathbf{\Phi} - \operatorname{div}\sigma(\mathbf{\Phi},\psi) - (\mathbf{u}_s\cdot\nabla)\mathbf{\Phi} + (\nabla\mathbf{u}_s)^\top\mathbf{\Phi} = \mathbf{G}_f \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{\Phi} = 0 \ \text{ in } \Omega, \\ \mathbf{\Phi} = \zeta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_d\setminus\Gamma_s, \quad \sigma(\mathbf{\Phi},\psi)\mathbf{n} + \mathbf{u}_s\cdot\mathbf{n}\mathbf{\Phi} = 0 \ \text{ on } \Gamma_n, \\ \lambda\zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm}A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*\psi = G_s^1 \ \text{ in } (0,\ell_s), \\ \lambda\zeta_2 + \alpha\Delta_s^2\zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^*\mathbf{\Phi} = -\gamma_s^+\psi + \gamma_s^-\psi + G_s^2 \ \text{ in } (0,\ell_s), \\ \zeta_1(0) = \partial_x\zeta_1(0) = 0 \ \text{ and } \ \partial_x^2\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_2(\ell_s), \ \ \partial_x^3\zeta_1(\ell_s) = \dfrac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s). \end{cases} \tag{5.9.38}$$

Fluid equations

After integrating by parts, we get

$$\begin{aligned} &\Big\langle \mathbf{F}_f, \mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} \\ =&\Big\langle \lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},q) + (\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s - A_1\eta_1 - A_2\eta_2, \mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} \\ =&\Big\langle \mathbf{G}_f, \mathbf{v}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} - \int_{\Gamma_s}\sigma(\mathbf{v},q)\mathbf{n}\cdot\mathbf{\Phi} + \int_{\Gamma_s}\sigma(\mathbf{\Phi},\psi)\mathbf{n}\cdot\mathbf{v} \\ -&\Big\langle A_1\eta_1, \mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} - \Big\langle A_2\eta_2, \mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega),\mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} + \int_\Omega \psi\operatorname{div}\mathbf{v}. \end{aligned} \tag{5.9.39}$$

The previous integration by parts identities hold for all $\mathbf{v}, \mathbf{\Phi} \in \mathbf{H}^2(\Omega)$ and $q, \psi \in H^1(\Omega)$. Nevertheless, by a density argument, it can be shown that they remain valid for all $\mathbf{v}, \mathbf{\Phi} \in \mathbf{H}^{\frac{3}{2}+\alpha}(\Omega)$ and $q, \psi \in H^{\frac{1}{2}+\alpha}(\Omega)$.

Let us compute the terms $\underbrace{-\int_{\Gamma_s}\sigma(\mathbf{v},q)\mathbf{n}\cdot\mathbf{\Phi}}_{I_1}$ and $\underbrace{\int_{\Gamma_s}\sigma(\mathbf{\Phi},\psi)\mathbf{n}\cdot\mathbf{v}}_{I_2}$.

- $I_1 = \underbrace{-\int_{\Gamma_s^+}\sigma(\mathbf{v},q)\mathbf{n}^+\cdot\mathbf{\Phi}}_{I_{11}} \underbrace{-\int_{\Gamma_s^-}\sigma(\mathbf{v},q)\mathbf{n}^-\cdot\mathbf{\Phi}}_{I_{12}} \underbrace{-\int_{\Gamma_s^\ell}\sigma(\mathbf{v},q)\mathbf{n}^\ell\cdot\mathbf{\Phi}}_{I_{13}}$

$\underline{I_{11}}$. Since $\mathbf{n}^+ = -\vec{\mathbf{e}}_2$ and $\boldsymbol{\Phi} = (0, \zeta_2)^\top$ on Γ_s^+, we get

$$\begin{aligned}\sigma(\mathbf{v}, q)\mathbf{n}^+ \cdot \boldsymbol{\Phi} &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \zeta_2 \end{pmatrix} \\ &= -\sigma_{22}\zeta_2 \\ &= q\zeta_2 - 2\nu v_{2,y}\zeta_2.\end{aligned}$$

Thus,

$$\begin{aligned}I_{11} &= -\int_{\Gamma_s^+} q\zeta_2 + 2\nu \int_{\Gamma_s^+} v_{2,y}\zeta_2 \\ &= -\int_0^{\ell_s} [\gamma_s^+ q]\zeta_2 + 2\nu \int_0^{\ell_s} [\gamma_s^+ v_{2,y}]\zeta_2\end{aligned}$$

But, since $(v_{1,x} + v_{2,y})|_{\Gamma_s^+} = \gamma_s^+ A_3\eta_1$ and $v_1 = 0$ on Γ_s^+, we obtain

$$I_{11} = -\int_0^{\ell_s} [\gamma_s^+ q]\zeta_2 + 2\nu \int_0^{\ell_s} [\gamma_s^+ A_3\eta_1]\zeta_2.$$

$\underline{I_{12}}$. Since $\mathbf{n}^- = \vec{\mathbf{e}}_2 = -\mathbf{n}^+$,

$$I_{12} = \int_0^{\ell_s} [\gamma_s^- q]\zeta_2 - 2\nu \int_0^{\ell_s} [\gamma_s^- A_3\eta_1]\zeta_2.$$

$\underline{I_{13}}$. Since $\mathbf{n}^\ell = -\vec{\mathbf{e}}_1$ and $\Phi_2 = \zeta_2$ on Γ_s^ℓ, we have

$$\begin{aligned}\sigma(\mathbf{v}, q)\mathbf{n}^\ell \cdot \boldsymbol{\Phi} &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \zeta_2 \end{pmatrix} \\ &= -\sigma_{21}\zeta_2 \\ &= -\nu(v_{1,y} + v_{2,x})\zeta_2.\end{aligned}$$

Then, since $v_1|_{\Gamma_s^{\ell_s}} = 0$, we get

$$I_{13} = \nu \int_{\Gamma_s^\ell} v_{2,x}\zeta_2.$$

Therefore,

$$I_1 = -\int_0^{\ell_s} [\gamma_s^{+,-} q]\zeta_2 + 2\nu \int_0^{\ell_s} [\gamma_s^{+,-} A_3\eta_1]\zeta_2 + \nu \int_{\Gamma_s^\ell} v_{2,x}\zeta_2. \tag{5.9.40}$$

- $I_2 = \underbrace{\int_{\Gamma_s^+} \sigma(\boldsymbol{\Phi}, \psi)\mathbf{n} \cdot \mathbf{v}}_{I_{21}} + \underbrace{\int_{\Gamma_s^-} \sigma(\boldsymbol{\Phi}, \psi)\mathbf{n} \cdot \mathbf{v}}_{I_{22}} + \underbrace{\int_{\Gamma_s^{\ell_s}} \sigma(\boldsymbol{\Phi}, \psi)\mathbf{n} \cdot \mathbf{v}}_{I_{23}}$

$\underline{I_{21}}$. Since $\mathbf{n}^+ = -\vec{\mathbf{e}}_2$, we have

$$I_{21} = \int_0^{\ell_s} [\gamma_s^+ \psi]\eta_2 - 2\nu \int_0^{\ell_s} [\gamma_s^+ \Phi_{2,y}]\eta_2.$$

But, since $(\Phi_{1,x} + \Phi_{2,y})|_{\Gamma_s^+} = 0$ and $\Phi_1 = 0$ on Γ_s^+, we obtain

$$I_{21} = -\int_0^{\ell_s} [\gamma_s^+ \psi]\eta_2$$

$\underline{I_{22}}$. Similarly,

$$\int_{\Gamma_s^-} \sigma(\boldsymbol{\Phi}, \psi)\mathbf{n}^+ \cdot \mathbf{v} = \int_0^{\ell_s} [\gamma_s^- \psi]\eta_2.$$

$\underline{I_{23}}$. Since $\mathbf{n}^\ell = -\vec{\mathbf{e}}_1$ and $v_2 = \eta_2$ on Γ_s^ℓ, we have

$$\begin{aligned}\sigma(\mathbf{\Phi}, \psi)\mathbf{n}^\ell \cdot \mathbf{v} &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \\ &= -\sigma_{21}\eta_2 \\ &= -(\nu v_{2,x} + \nu v_{1,y})\eta_2,\end{aligned}$$

Then, from the equality $v_1 = 0$ on $\Gamma_s^{\ell_s}$, we get

$$I_{23} = -\nu \int_{\Gamma_s^\ell} \Phi_{2,x}\eta_2.$$

Therefore,

$$I_2 = \int_0^{\ell_s} [\gamma_s^{+,-}\psi]\eta_2 - \nu \int_{\Gamma_s^\ell} \Phi_{2,x}\eta_2, \tag{5.9.41}$$

and thus,

$$\begin{aligned}I_1 + I_2 = &- \int_0^{\ell_s} [\gamma_s^{+,-} q]\zeta_2 + 2\nu \int_0^{\ell_s} [\gamma_s^{+,-} A_3\eta_1]\zeta_2 + \nu \int_{\Gamma_s^\ell} v_{2,x}\zeta_2 \\ &+ \int_0^{\ell_s} [\gamma_s^{+,-}\psi]\eta_2 - \nu \int_{\Gamma_s^\ell} \Phi_{2,x}\eta_2.\end{aligned} \tag{5.9.42}$$

Substituting (5.9.42) into (5.9.39), we get

$$\begin{aligned}&\Big\langle \mathbf{F}_f, \mathbf{\Phi} \Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega), \mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} \\ =&\Big\langle \mathbf{G}_f, \mathbf{v} \Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega), \mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} - \int_0^{\ell_s} [\gamma_s^{+,-} q]\zeta_2 + 2\nu \int_0^{\ell_s} [\gamma_s^{+,-} A_3\eta_1]\zeta_2 \\ &+ \nu \int_{\Gamma_s^\ell} v_{2,x}\zeta_2 + \int_0^{\ell_s} [\gamma_s^{+,-}\psi]\eta_2 - \nu \int_{\Gamma_s^\ell} \Phi_{2,x}\eta_2 \\ &- \Big\langle A_1\eta_1, \mathbf{\Phi} \Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega), \mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} - \Big\langle A_2\eta_2, \mathbf{\Phi} \Big\rangle_{\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega), \mathbf{H}^{\frac{1}{2}-\alpha}(\Omega)} + \int_\Omega \psi A_3\eta_1.\end{aligned} \tag{5.9.43}$$

Structure equations

In the computations developed in this subsection, we consider the space $H^2_{\{0\}}(0, \ell_s)$ endowed with the inner product

$$\langle \eta, \mu \rangle_{H^2_{\{0\}}} = \alpha \int_0^{\ell_s} \Delta\eta \cdot \Delta\mu, \quad \text{for all } \eta, \mu \in H^2_{\{0\}}(0, \ell_s).$$

- Taking the inner product $\langle \cdot, \cdot \rangle_{H^2_{\{0\}}}$ of both sides of the equality

$$\lambda\zeta_1 + \zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}(A_4^* - 2\nu(\gamma_s^{\pm} A_3)^*)\zeta_2 - \frac{1}{\alpha}(\Delta_s)^{-1}A_1^*\mathbf{\Phi} + \frac{1}{\alpha}(\Delta_s)^{-1}A_3^*\psi = G_s^1,$$

with η_1, we get

$$\begin{aligned}&\alpha\lambda \int_0^{\ell_s} \Delta\zeta_1 \cdot \Delta\eta_1 + \alpha \int_0^{\ell_s} \Delta\zeta_2 \cdot \Delta\eta_1 - \int_0^{\ell_s} [A_4^*\zeta_2] \cdot \eta_1 + 2\nu \int_0^{\ell_s} [(\gamma_s^{+,-} A_3)^*\zeta_2] \cdot \eta_1 \\ &+ \int_0^{\ell_s} [A_1^*\mathbf{\Phi}] \cdot \eta_1 + \int_0^{\ell_s} [A_3^*\Psi] \cdot \eta_1 = \alpha \int_0^{\ell_s} \Delta G_s^1 \cdot \Delta\eta_1.\end{aligned} \tag{5.9.44}$$

• Applying the L^2 inner product with η_2 to both sides of the equality

$$\lambda\zeta_2 + \alpha\Delta_s^2\zeta_1 + \gamma(\Delta_s^2)^{\frac{1}{2}}\zeta_2 - A_2^*\mathbf{\Phi} = -\gamma_s^+\psi + \gamma_s^-\psi + G_s^2 \text{ in } (0,\ell_s),$$

we obtain

$$\begin{aligned}&\lambda\int_0^{\ell_s}\zeta_2\eta_2 - \alpha\int_0^{\ell_s}\Delta\zeta_1\cdot\Delta\eta_2 + \gamma\int_0^{\ell_s}[(\Delta_s^2)^{\frac{1}{2}}\zeta_2]\cdot\eta_2 - \int_0^{\ell_s}A_2^*\mathbf{\Phi}\cdot\eta_1\\ =&\int_0^{\ell_s}G_s^2\eta_2 + \alpha\frac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s)\eta_2(\ell_s) - \alpha\frac{2\nu e}{\alpha}\zeta_2(\ell_s)\eta_{2,x}(\ell_s).\end{aligned} \tag{5.9.45}$$

• By taking the inner product $\langle\cdot,\cdot\rangle_{H^2_{\{0\}}}$ in both sides of the equality

$$\lambda\eta_1 - \eta_2 = F_s^1 \text{ on } (0,\ell_s),$$

with ζ_1, we get

$$\alpha\lambda\int_0^{\ell_s}\Delta\eta_1\cdot\Delta\zeta_1 - \alpha\int_0^{\ell_s}\Delta\eta_2\cdot\Delta\zeta_1 = \alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1. \tag{5.9.46}$$

• Applying the L^2 inner product with ζ_2 to both sides of the equality

$$\lambda\eta_2 + \alpha\Delta^2\eta_1 + \gamma(\Delta_s^2)^{1/2}\eta_2 - A_4\eta_1 = -\gamma_s^+q + \gamma_s^-q + F_s^2,$$

we obtain

$$\begin{aligned}&\lambda\int_0^{\ell_s}\eta_2\zeta_2 + \alpha\int_0^{\ell_s}\Delta\eta_1\cdot\Delta\zeta_2 + \gamma\int_0^{\ell_s}[(\Delta_s^2)^{1/2}\eta_2]\zeta_2 - \int_0^{\ell_s}[A_4\eta_1]\zeta_2\\ =&-\int_0^{\ell_s}[\gamma_s^+q]\zeta_2 + \int_0^{\ell_s}[\gamma_s^-q]\zeta_2 + \int_0^{\ell_s}F_s^2\zeta_2.\end{aligned} \tag{5.9.47}$$

By adding the identities (5.9.44), (5.9.45), (5.9.46) and (5.9.47), we obtain

$$\begin{aligned}&\alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1 - \int_0^{\ell_s}[\gamma_s^+q]\zeta_2 + \int_0^{\ell_s}[\gamma_s^-q]\zeta_2 + \int_0^{\ell_s}F_s^2\zeta_2\\ &\alpha\lambda\int_0^{\ell_s}\Delta\zeta_1\cdot\Delta\eta_1 + \alpha\int_0^{\ell_s}\Delta\zeta_2\cdot\Delta\eta_1 - \int_0^{\ell_s}[A_4^*\zeta_2]\cdot\eta_1 + 2\nu\int_0^{\ell_s}[(\gamma_s^{+,-}A_3)^*\zeta_2]\cdot\eta_1\\ &+\int_0^{\ell_s}[A_1^*\mathbf{\Phi}]\cdot\eta_1 + \int_0^{\ell_s}[A_3^*\Psi]\cdot\eta_1 + \lambda\int_0^{\ell_s}\zeta_2\eta_2 - \alpha\int_0^{\ell_s}\Delta\zeta_1\cdot\Delta\eta_2\\ &+\gamma\int_0^{\ell_s}[(\Delta_s^2)^{1/2}\zeta_2]\cdot\eta_2 - \int_0^{\ell_s}A_2^*\mathbf{\Phi}\cdot\eta_1\\ =&\alpha\lambda\int_0^{\ell_s}\Delta\eta_1\cdot\Delta\zeta_1 - \alpha\int_0^{\ell_s}\Delta\eta_2\cdot\Delta\zeta_1\\ &\lambda\int_0^{\ell_s}\eta_2\zeta_2 + \alpha\int_0^{\ell_s}\Delta\eta_1\cdot\Delta\zeta_2 + \gamma\int_0^{\ell_s}[(\Delta_s^2)^{1/2}\eta_2]\zeta_2 - \int_0^{\ell_s}[A_4\eta_1]\zeta_2 + \alpha\int_0^{\ell_s}\Delta G_s^1\cdot\Delta\eta_1\\ &+\int_0^{\ell_s}G_s^2\eta_2 + \alpha\frac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s)\eta_2(\ell_s) - \alpha\frac{2\nu e}{\alpha}\zeta_2(\ell_s)\eta_{2,x}(\ell_s),\end{aligned} \tag{5.9.48}$$

or equivalently

$$
\begin{aligned}
&\alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1-\int_0^{\ell_s}[\gamma_s^+q]\zeta_2+\int_0^{\ell_s}[\gamma_s^-q]\zeta_2+\int_0^{\ell_s}F_s^2\zeta_2\\
&-\int_0^{\ell_s}[A_4^*\zeta_2]\cdot\eta_1+2\nu\int_0^{\ell_s}[(\gamma_s^{+,-}A_3)^*\zeta_2]\cdot\eta_1\\
&+\int_0^{\ell_s}[A_1^*\mathbf{\Phi}]\cdot\eta_1+\int_0^{\ell_s}[A_3^*\Psi]\cdot\eta_1-\int_0^{\ell_s}A_2^*\mathbf{\Phi}\cdot\eta_1\\
=&-\int_0^{\ell_s}[A_4\eta_1]\zeta_2+\alpha\int_0^{\ell_s}\Delta G_s^1\cdot\Delta\eta_1+\int_0^{\ell_s}G_s^2\eta_2+\alpha\frac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s)\eta_2(\ell_s)-\alpha\frac{2\nu e}{\alpha}\zeta_2(\ell_s)\eta_{2,x}(\ell_s).
\end{aligned}
\tag{5.9.49}
$$

Finally, after adding up identities (5.9.43) and (5.9.49), we get

$$
\begin{aligned}
&\Big\langle\mathbf{F}_f,\mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}+\alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1-\int_0^{\ell_s}[\gamma_s^+q]\zeta_2+\int_0^{\ell_s}[\gamma_s^-q]\zeta_2+\int_0^{\ell_s}F_s^2\zeta_2\\
&-\int_0^{\ell_s}[A_4^*\zeta_2]\cdot\eta_1+2\nu\int_0^{\ell_s}[(\gamma_s^{+,-}A_3)^*\zeta_2]\cdot\eta_1\\
&+\int_0^{\ell_s}[A_1^*\mathbf{\Phi}]\cdot\eta_1+\int_0^{\ell_s}[A_3^*\Psi]\cdot\eta_1-\int_0^{\ell_s}[A_2^*\mathbf{\Phi}]\cdot\eta_1\\
=&\Big\langle\mathbf{G}_f,\mathbf{v}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}-\int_0^{\ell_s}[\gamma_s^{+,-}q]\zeta_2+2\nu\int_0^{\ell_s}[\gamma_s^{+,-}A_3\eta_1]\zeta_2\\
&+\nu\int_{\Gamma_s^\ell}v_{2,x}\zeta_2+\int_0^{\ell_s}[\gamma_s^{+,-}\psi]\eta_2-\nu\int_{\Gamma_s^\ell}\Phi_{2,x}\eta_2\\
&-\Big\langle A_1\eta_1,\mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}-\Big\langle A_2\eta_2,\mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}+\int_\Omega\psi A_3\eta_1\\
&-\int_0^{\ell_s}[A_4\eta_1]\zeta_2+\alpha\int_0^{\ell_s}\Delta G_s^1\cdot\Delta\eta_1+\int_0^{\ell_s}G_s^2\eta_2+\alpha\frac{2\nu e}{\alpha}\zeta_{2,x}(\ell_s)\eta_2(\ell_s)-\alpha\frac{2\nu e}{\alpha}\zeta_2(\ell_s)\eta_{2,x}(\ell_s),
\end{aligned}
\tag{5.9.50}
$$

or equivalently,

$$
\begin{aligned}
&\Big\langle\mathbf{F}_f,\mathbf{\Phi}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}+\alpha\int_0^{\ell_s}\Delta F_s^1\cdot\Delta\zeta_1+\int_0^{\ell_s}F_s^2\zeta_2\\
=&\Big\langle\mathbf{G}_f,\mathbf{v}\Big\rangle_{\mathbf{H}^{-\frac12+\alpha}(\Omega),\mathbf{H}^{\frac12-\alpha}(\Omega)}+\alpha\int_0^{\ell_s}\Delta G_s^1\cdot\Delta\eta_1+\int_0^{\ell_s}G_s^2\eta_2.
\end{aligned}
\tag{5.9.51}
$$

Appendix D: Proof of Proposition 5.7.1

Let $\widetilde{\mathbf{v}} = P\mathbf{v} + \nabla N_{\mathrm{div}} A_3 \eta_1 - \nabla N_s \eta_2$ and $\widetilde{\mathbf{\Phi}} = P\mathbf{\Phi} - \nabla N_s \zeta_2$, with $(P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{Z}_0$ and $(P\mathbf{\Phi}, \zeta_1, \zeta_2) \in \mathbf{Z}_0^*$. Then,

$$\begin{aligned}\left\langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{\Phi}} \right\rangle_{\mathbf{L}^2(\Omega)} = \left\langle P\mathbf{v}, P\mathbf{\Phi} \right\rangle_{\mathbf{L}^2(\Omega)} & \underbrace{- \int_\Omega P\mathbf{v} \cdot \nabla N_s \zeta_2}_{I_1} + \underbrace{\int_\Omega \nabla N_{\mathrm{div}} A_3 \eta_1 \cdot P\mathbf{\Phi}}_{I_2} \\ & \underbrace{- \int_\Omega \nabla N_{\mathrm{div}} A_3 \eta_1 \cdot \nabla N_s \zeta_2}_{I_3} \underbrace{- \int_\Omega \nabla N_s \eta_2 \cdot P\mathbf{\Phi}}_{I_4} + \underbrace{\int_\Omega \nabla N_s \eta_2 \cdot \nabla N_s \zeta_2}_{I_5} .\end{aligned} \tag{5.9.52}$$

We now compute each term of the preceding identity.

- $\underline{I_1}$

Let us first notice that $q = N_s \zeta_2$ satisfies

$$\begin{cases} \Delta q = 0 \ \text{ in } \Omega, \ \dfrac{\partial q}{\partial \mathbf{n}} = \zeta_2 \text{ on } \Gamma_s^+, \ \dfrac{\partial q}{\partial \mathbf{n}} = -\zeta_2 \text{ on } \Gamma_s^-, \\ \dfrac{\partial q}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \ q = 0 \text{ on } \Gamma_n. \end{cases} \tag{5.9.53}$$

Then,

$$I_1 = -\int_\Omega P\mathbf{v} \cdot \nabla N_s \zeta_2 = \int_\Omega \mathrm{div}(P\mathbf{v}) q - \int_\Gamma q P\mathbf{v} \cdot \mathbf{n} = 0. \tag{5.9.54}$$

- $\underline{I_2}$

Since $q = N_{\mathrm{div}} A_3 \eta_1$ satisfies

$$\Delta q = A_3 \eta_1 \ \text{ in } \Omega, \ \frac{\partial q}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \ \ q = 0 \ \text{ on } \Gamma_n, \tag{5.9.55}$$

then,

$$I_2 = -\int_\Omega P\mathbf{v} \cdot \nabla N_s \zeta_2 = \int_\Omega \mathrm{div}(P\mathbf{v}) q - \int_\Gamma q P\mathbf{v} \cdot \mathbf{n} = 0. \tag{5.9.56}$$

- $\underline{I_3}$

Let us first notice that $q = N_{\mathrm{div}} A_3 \eta_1$ and $\widetilde{q} = N_s \zeta_2$ satisfy

$$\Delta q = A_3 \eta_1 \ \text{ in } \Omega, \ \frac{\partial q}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \ \ q = 0 \ \text{ on } \Gamma_n, \tag{5.9.57}$$

and

$$\begin{cases} \Delta \widetilde{q} = 0 \ \text{ in } \Omega, \ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = \zeta_2 \text{ on } \Gamma_s^+, \ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = -\zeta_2 \text{ on } \Gamma_s^-, \\ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \ \widetilde{q} = 0 \text{ on } \Gamma_n, \end{cases} \tag{5.9.58}$$

respectively. Let us now observe that

$$\begin{aligned} I_3 &= -\int_\Omega \nabla q \, \widetilde{q} - \int_\Gamma \frac{\partial q}{\partial \mathbf{n}} \widetilde{q} \\ &= \int_\Omega (A_3 \eta_1)(N_s \zeta_2) \\ &= \left\langle \eta_1, \frac{1}{\alpha} (\Delta_s)^{-1} A_3^* N_s \zeta_2 \right\rangle_{H^2_{\{0\}}(0,\ell_s),(H^2_{\{0\}}(0,\ell_s))'} . \end{aligned} \tag{5.9.59}$$

• $\underline{I_4}$

The calculations are similar to those performed to compute I_1 in (5.9.54). Thus,

$$I_4 = 0. \tag{5.9.60}$$

• $\underline{I_5}$

We first notice that $q = N_s\eta_2$ and $\widetilde{q} = N_s\zeta_2$ are the solutions to

$$\begin{cases} \Delta q = 0 \ \text{ in } \Omega, \ \dfrac{\partial q}{\partial \mathbf{n}} = \eta_2 \text{ on } \Gamma_s^+, \ \dfrac{\partial q}{\partial \mathbf{n}} = -\eta_2 \text{ on } \Gamma_s^-, \\ \dfrac{\partial q}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \ q = 0 \text{ on } \Gamma_n. \end{cases} \tag{5.9.61}$$

and

$$\begin{cases} \Delta \widetilde{q} = 0 \ \text{ in } \Omega, \ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = \zeta_2 \text{ on } \Gamma_s^+, \ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = -\zeta_2 \text{ on } \Gamma_s^-, \\ \dfrac{\partial \widetilde{q}}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_d \setminus (\Gamma_s^- \cup \Gamma_s^+), \ \widetilde{q} = 0 \text{ on } \Gamma_n, \end{cases} \tag{5.9.62}$$

respectively. Then,

$$I_5 = \int_\Omega \nabla q \cdot \nabla \widetilde{q} = -\int_\Omega \Delta q \, \widetilde{q} + \int_\Gamma \frac{\partial q}{\partial \mathbf{n}} \widetilde{q} = \left\langle \eta_2, \gamma_s^{+,-} N_s \zeta_2 \right\rangle_{L^2(0,\ell_s)}. \tag{5.9.63}$$

Thus, using (5.9.54), (5.9.56), (5.9.59), (5.9.60) and (5.9.63) in the identity (5.9.52), we get

$$\begin{aligned} \left\langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{\Phi}} \right\rangle_{\mathbf{L}^2(\Omega)} = \left\langle P\mathbf{v}, P\mathbf{\Phi} \right\rangle_{\mathbf{L}^2(\Omega)} &+ \left\langle \eta_1, \frac{1}{\alpha} (\Delta_s)^{-1} A_3^* N_s \zeta_2 \right\rangle_{H^2_{\{0\}}(0,\ell_s),(H^2_{\{0\}}(0,\ell_s))'} \\ &+ \left\langle \eta_2, \gamma_s^{+,-} N_s \zeta_2 \right\rangle_{L^2(0,\ell_s)}. \end{aligned} \tag{5.9.64}$$

Hence, after adding

$$\left\langle \eta_1, \zeta_1 \right\rangle_{H^2_{\{0\}}(0,\ell_s),(H^2_{\{0\}}(0,\ell_s))'} + \left\langle \eta_2, \zeta_2 \right\rangle_{L^2(0,\ell_s)}$$

to both sides of identity (5.9.64), we obtain

$$\begin{aligned} \left\langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{\Phi}} \right\rangle_{\mathbf{L}^2(\Omega)} = \left\langle P\mathbf{v}, P\mathbf{\Phi} \right\rangle_{\mathbf{L}^2(\Omega)} &+ \left\langle \eta_1, \zeta_1 \frac{1}{\alpha} (\Delta_s)^{-1} A_3^* N_s \zeta_2 \right\rangle_{H^2_{\{0\}}(0,\ell_s),(H^2_{\{0\}}(0,\ell_s))'} \\ &+ \left\langle \eta_2, \zeta_2 + \gamma_s^{+,-} N_s \zeta_2 \right\rangle_{L^2(0,\ell_s)}. \end{aligned} \tag{5.9.65}$$

Finally, using the expression of M^* (see Proposition 5.4.1) we conclude the proof of Proposition 5.7.1.

Appendix E: Proof of Proposition 5.7.3

Let us notice that for all $j = 1, \dots, N_u$, we have

$$\begin{aligned}
&\overbrace{\frac{d}{dt}\left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'}}^{J_1} = \overbrace{\left\langle \Pi_u \mathcal{A} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'}}^{J_2} \\
&+ \underbrace{\omega \left\langle \Pi_u \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'}}_{J_3} + \underbrace{\left\langle \Pi_u \mathcal{B}\mathbf{f}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'}}_{J_4}.
\end{aligned} \tag{5.9.66}$$

Let us analyze each of the terms appearing in the previous identity.

- $\underline{J_1}$:

$$\begin{aligned}
J_1 &= \frac{d}{dt}\left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\
&= \frac{d}{dt}\left\langle \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\
&= \frac{d}{dt}\left\langle \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\
&= \frac{d}{dt}\sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \left\langle \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\
&= \frac{d}{dt}\sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \delta_{i,j}.
\end{aligned} \tag{5.9.67}$$

- $\underline{J_2}$:

$$\begin{aligned}
J_2 &= \left\langle \Pi_u \mathcal{A} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\
&= \left\langle \mathcal{A}\Pi_u \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\
&= \left\langle \mathcal{A} \left[\sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix} \right], M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\
&= \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \left\langle \mathcal{A} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\boldsymbol{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'}.
\end{aligned} \tag{5.9.68}$$

• $\underline{J_3}$:

$$\begin{aligned} J_3 &= \omega \left\langle \Pi_u \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\ &= \omega \left\langle \Pi_u \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \omega \left\langle \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \left\langle \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \delta_{i,j}. \end{aligned} \tag{5.9.69}$$

• $\underline{J_4}$:

$$\begin{aligned} J_4 &= \left\langle \Pi_u \mathcal{B}\mathbf{f}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z},\mathbf{Z}'} \\ &= \left\langle \Pi_u \mathcal{B}\mathbf{f}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \left\langle \Pi_u \left[\sum_{k=1}^{N_c} f_k \begin{pmatrix} 0 \\ 0 \\ (I+\gamma_s^{+,-} N_s)^{-1} w_k \end{pmatrix} \right], M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \sum_{k=1}^{N_c} \left\langle f_k \sum_{i=1}^{N_u} \left\langle \begin{pmatrix} 0 \\ 0 \\ (I+\gamma_s^{+,-} N_s)^{-1} w_k \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_i \\ \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \sum_{k=1}^{N_c} f_k \sum_{i=1}^{N_u} w_k \zeta_{2,i} \left\langle \begin{pmatrix} P\mathbf{v}_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \\ &= \sum_{k=1}^{N_c} f_k \sum_{i=1}^{N_u} w_k \zeta_{2,i} \delta_{i,j}. \end{aligned} \tag{5.9.70}$$

Then, by setting

$$\zeta_u = \left(\left\langle \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1 \le j \le N_u},$$

we observe from identities (5.9.67), (5.9.68), (5.9.69) and (5.9.70) that system (5.9.66) can be rewritten as follows:

$$\zeta_u' = (\Lambda_u + \omega I)\zeta_u + \mathcal{B}_u \mathbf{f}, \quad \zeta_u(0) = \zeta_u^0, \tag{5.9.71}$$

where

$$\zeta_u^0 = \left(\left\langle \begin{pmatrix} P\mathbf{v}^0 \\ 0 \\ \eta_2 \end{pmatrix}, M^* \begin{pmatrix} P\mathbf{\Phi}_j \\ \zeta_{1,j} \\ \zeta_{2,j} \end{pmatrix} \right\rangle_{\mathbf{Z}_0} \right)_{1 \le j \le N_u}.$$

The second equation in (5.7.7) is obtained by applying the operator Π_s to both sides of the first equation in (5.7.5). This completes the proof.

Chapter 6

Numerical simulations of the stabilization problem of the fluid-structure interaction system

Abstract of the current chapter

In this chapter, we study the numerical stabilization of the fluid-structure interaction system. More precisely, we study the stabilization of the system corresponding to the semi-discretization in space of the fluid-structure interaction system. Numerical simulations for different parameter values are presented.

Outline of the current chapter

6.1 Introduction 210

6.2 Stabilization of the semi-discrete system 213

6.2.1 Semi-discrete approximation . . . 214

6.2.2 Eliminating the fluid domain variables $\overline{\mathbf{W}}$ and $\overline{\mathbf{D}}$ . . . 216

6.2.3 Finite dimensional controlled system . . . 217

6.2.4 Spectral decomposition of the operators . . . 217

6.2.5 The projected dynamical system and computation of the feedback 219

6.3 Algorithm for FSI direct problem 220

6.4 Numerical experiments 221

6.4.1 Case 1: Rigidity coefficient $\boldsymbol{\alpha = 1}$, $Re = 200$ and $\gamma = 10^{-6}$ . . . 223

6.4.2 Case 2: Rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, $Re = 200$ and $\gamma = 10^{-6}$ . . 228

6.4.3 Case 3: Rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, $Re = 200$ and $\gamma = 10^{-6}$ . . 233

6.1 Introduction

Thoughout this chapter, the notation used to describe the reference domain Ω and the physical domain $\Omega_{\eta(t)}$, as well as their boundaries, will be the same as that introduced in Subsection 3.1.1 of Chapter 3.

Let us first recall the stabilization problem. Let $(\mathbf{u}_s, p_s)$ be a solution of the stationary Navier-Stokes equations

$$\begin{cases} (\mathbf{u}_s\cdot\nabla)\mathbf{u}_s - \operatorname{div}\sigma(\mathbf{u}_s,p_s) = 0, \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{u}_s = 0 \ \text{ in } \Omega, \\ \mathbf{u}_s = \mathbf{g}_s^i, \ \text{ on } \Gamma_i, \ \ \mathbf{u}_s = 0 \ \text{ on } \Gamma\setminus\Gamma_i, \\ \sigma(\mathbf{u}_s,p_s)\mathbf{n} = 0 \ \text{ on } \Gamma_n. \end{cases} \tag{6.1.1}$$

Let us consider the system

$$\begin{cases} \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q_\eta^\infty, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ in } Q_\eta^\infty, \\ \mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma_i^\infty, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma_w^\infty, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma_r^\infty, \\ \mathbf{u} = \eta_t\vec{\mathbf{e}}_2 \ \text{ on } \Sigma_\eta^\infty, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma_n^\infty, \\ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega, \\ \partial_t^2\eta + \alpha\Delta_s^2\eta + \gamma B\eta_t = H(\mathbf{u},p,\eta) + f_s + f \text{ in } (0,\infty)\times(0,\ell_s), \\ \eta = 0 \ \text{ and } \ \partial_{x_1}\eta = 0 \ \text{ on } (0,\infty)\times\{0\}, \\ \partial_{x_1}^2\eta = 0 \ \text{ and } \ \partial_{x_1}^3\eta = 0 \ \text{ on } (0,\infty)\times\{\ell_s\}, \\ \eta(0) = 0 \ \text{ and } \ \partial_t\eta(0) = \eta_2^0 \ \text{ in } (0,\ell_s), \end{cases} \tag{6.1.2}$$

where $\mathbf{u}$ and p respresent the fluid velocity and pressure. The Cauchy stress tensor $\sigma(\mathbf{u},p)$ is given by

$$\sigma(\mathbf{u},p) = 2\nu\varepsilon(\mathbf{u}) - pI, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top),$$

with $\nu > 0$ denoting the fluid viscosity. Here, the inflow boundary condition $\mathbf{g}^i = \mathbf{g}_s^i + \beta\mathbf{g}_p^i$, where $\mathbf{g}_s^i$ is assumed to be time independent, $\mathbf{g}_p^i$ is a time dependent perturbation, while β denotes the amplitude perturbation. The parameters $\alpha > 0$ and $\gamma > 0$ are constants relative to the structure. The damping operator B is given by

$$B = \Delta_s^2 = \partial_{x_1}^4, \quad D(B) = H^4_{\{0,\ell_s\}}(0,\ell_s).$$

The expression of the force exerted by the fluid on $\Gamma_{\eta(t)}^+ \cup \Gamma_{\eta(t)}^-$ is given by

$$H(\mathbf{u},p,\eta) = -\left(\sigma^+(\mathbf{u},p)\mathbf{n}_{\eta(t)}^+ + \sigma^-(\mathbf{u},p)\mathbf{n}_{\eta(t)}^-\right)\sqrt{1+(\partial_{x_1}\eta)^2}\cdot\vec{\mathbf{e}}_2, \tag{6.1.3}$$

where

$$\sigma^\pm(\mathbf{u},p) = \sigma(\mathbf{u}(t,x,\eta(t,x)\pm e), p(t,x,\eta(t,x)\pm e)),$$

and $n_{\eta(t)}^+$ (resp. $n_{\eta(t)}^+$) is the unit normal to $\Gamma_{\eta(t)}^+$ (resp. $\Gamma_{\eta(t)}^-$) exterior to $\Omega_{\eta(t)}$.

The function f_s is time-independent and is chosen in such a way that the triplet $(\mathbf{u},\eta,\eta_t) = (\mathbf{u}_s,0,0)$ constitutes a stationary solution of system (6.1.2). Thus,

$$f_s = \sigma(\mathbf{u}_s,p_s)\mathbf{n}^+\cdot\vec{\mathbf{e}}_2 + \sigma(\mathbf{u}_s,p_s)\mathbf{n}^-\cdot\vec{\mathbf{e}}_2, \tag{6.1.4}$$

where $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) represents the outward unit normal to Γ_s^+ (resp. Γ_s^-).

The control f take the form

$$f = \sum_{j=1}^{N_c} f_j(t) w_j(z_1), \tag{6.1.5}$$

where the family $(w_j)_{1 \le j \le N_c}$ is defined in (5.7.3).

The objective is to find a control $\mathbf{f} = (f_1, \ldots, f_{N_c})$ given in a feedback form, able to stabilize, with any decay rate $\omega \geq 0$, the system (6.1.2) around the stationary solution $(\mathbf{u}_s, 0, 0)$, provided $\mathbf{g}_p^i$, $\mathbf{u}^0 - \mathbf{u}_s$ and η_2^0 are small enough in appropriate funcional spaces.

The starting point of the analysis lies in the ideas presented in [FNR], where the authors study the numerical stabilization problem of a fluid-structure interaction model. In that work, the strategy used is similar to the one presented in Chapter 5: we first seek a feedback law that stabilizes the semi-discretized system in space, and then prove that this feedback law also locally stabilizes the nonlinear system, under certain smallness conditions on the initial and boundary data. Two important elements underlie the analysis carried out in [FNR]:

(1) The first important aspect is that, to rewrite the fluid-structure system in the reference domain, the authors employ an explicit geometric transformation, similar to the one defined in [FNR19] or the one used Chapter 3. Then, the linearization of the resulting system is performed "manually." Once this step is completed, the calculation of the feedback law aimed at stabilizing the linearized system is performed in the reference domain Ω.

(2) The second important point is that, once the feedback law for the linearized system is computed in the fixed reference domain Ω, it is injected into the nonlinear system, which is then solved in the reference domain.

In summary, the numerical stabilization analysis carried out in the cited paper [FNR] was conducted entirely within a fixed reference framework Ω.

Let us now compare the strategy adopted in [FNR] with the one employed in this chapter.

(1b) To rewrite the fluid-structure interaction system in the reference domain, we do not employ an explicit geometric transformation as in [FNR]. Instead, we consider a transformation defined in terms of the harmonic extension over Ω of the trace of the structure's displacement. Then, once the nonlinear system has been reformulated in its weak form on the fixed reference domain Ω, the linearization is carried out using a routine provided by GetFEM++ library. Next, and similarly to the approach in [FNR], the feedback law that stabilizes the system is computed on the fixed reference domain Ω.

(2b) Analogously to the approach adopted in [FNR], the feedback law computed on the fixed reference domain Ω is applied to the nonlinear system, which, unlike in [FNR], is solved on the moving physical domain $\Omega_{\eta(t)}$ by using the semi-implicit algorithm 3.

In summary, the two main differences between the strategies $(1b) - (2b)$ used in this chapter compared to those implemented in [FNR] $(1) - (2)$ are the following. Firstly, the tranformations employed to rewrite the system in the reference configuration are not the same. Secondly, the approach used to solve the direct problem differs in both cases. Regarding the previous remark, it is important to highlight that solving the direct problem in the actual fluid domain is an approach that has also been employed, for instance, in [Del18].

Let us now recall the variational formulation of system (6.1.2) introduced in Section 4.2 of

Chapter 4. By setting $\eta_1 := \eta$ and $\eta_2 := \eta_{1,t}$, system (6.1.2) takes the form

$$\left\{\begin{array}{l}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0 \ \text{ in } Q^\infty_{\eta_1},\\
\operatorname{div}\mathbf{u} = 0 \ \text{ in } Q^\infty_{\eta_1},\\
\mathbf{u} = \mathbf{g}^i \ \text{ on } \Sigma^\infty_i, \ \ \mathbf{u} = 0 \ \text{ on } \Sigma^\infty_r \cup \Sigma^\infty_w,\\
\mathbf{u} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Sigma^\infty_{\eta_1}, \ \ \sigma(\mathbf{u},p)\mathbf{n} = 0 \ \text{ on } \Sigma^\infty_n,\\
\mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega,\\
\eta_2 = \eta_{1,t} \text{ in } (0,\infty)\times(0,\ell_s),\\
\partial_t\eta_2 + \alpha\Delta_s^2\eta_1 + \gamma B\eta_2 = H(\mathbf{u},p,\eta_1) + f_s + f \text{ in } (0,\infty)\times(0,\ell_s),\\
\eta_1 = 0 \ \text{ and } \ \partial_{x_1}\eta_1 = 0 \ \text{ on } (0,T)\times\{0\},\\
\partial^2_{x_1}\eta_1 = 0 \ \text{ and } \ \partial^3_{x_1}\eta_1 = 0 \ \text{ on } (0,\infty)\times\{\ell_s\},\\
\eta_1(0) = 0 \ \text{ and } \ \partial_2\eta(0) = \eta_2^0 \ \text{ in } (0,\ell_s).
\end{array}\right. \tag{6.1.6}$$

We set $\Gamma_0 = \Gamma_r \cup \Gamma_w$. In order to take into account the Dirichlet boundary conditions on Γ_d, we introduce the Lagrange multiplier $\boldsymbol{\lambda}^u = (\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_{s,top}, \boldsymbol{\lambda}_{s,bot}, \boldsymbol{\lambda}_{s,lat})^\top$, where:

$$\begin{array}{rl}
\boldsymbol{\lambda}_i & \text{is the multiplier associated to the boundary data } \mathbf{g}^i \text{ prescribed on } \Gamma_i,\\
\boldsymbol{\lambda}_0 & \text{is the multiplier associated to the null boundary data prescribed on } \Gamma_r\cup\Gamma_w,\\
\boldsymbol{\lambda}_{s,top} & \text{is the multiplier associated to the kinematic condition imposed on } \Gamma^+_{\eta_1(t)},\\
\boldsymbol{\lambda}_{s,bot} & \text{is the multiplier associated to the kinematic condition imposed on } \Gamma^-_{\eta_1(t)},\\
\boldsymbol{\lambda}_{s,lat} & \text{is the multiplier associated to the kinematic condition imposed on } \Gamma^\ell_{\eta_1(t)}.
\end{array} \tag{6.1.7}$$

Then, the variational formulation of system (6.1.6) reads as follows:

Find $\eta_1,\eta_2 \in L^2_{loc}(0,\infty; H^2_{\{0\}}(0,\ell_s))$, $\mathbf{u} \in H^1_{loc}(0,\infty;\mathbf{H}^{-\frac12+\alpha}(\Omega_{\eta_1(t)})) \cap L^2_{loc}(0,\infty;\mathbf{H}^1(\Omega_{\eta_1(t)}))$, $p \in L^2_{loc}(0,\infty;L^2(\Omega_{\eta_1(t)}))$ and $\boldsymbol{\lambda}^u \in L^2_{loc}(0,\infty;\mathbf{H}^{-\frac12}(\Gamma_d))$ such that

$$\left\{\begin{array}{l}
\displaystyle\int_{\Omega_{\eta_1(t)}} \partial_t\mathbf{u}\cdot\boldsymbol{\phi} = a_f(\mathbf{u},\boldsymbol{\phi}) + b(\boldsymbol{\phi},p) + c(\mathbf{u},\mathbf{u},\boldsymbol{\phi}) + \int_{\Gamma_i}\boldsymbol{\lambda}_i\cdot\boldsymbol{\phi} + \int_{\Gamma_0}\boldsymbol{\lambda}_0\cdot\boldsymbol{\phi}\\
\displaystyle\qquad\qquad + \int_{\Gamma^+_{\eta_1(t)}}\boldsymbol{\lambda}_{s,top}\cdot\boldsymbol{\phi} + \int_{\Gamma^-_{\eta_1(t)}}\boldsymbol{\lambda}_{s,bot}\cdot\boldsymbol{\phi}, + \int_{\Gamma^\ell_{\eta_1(t)}}\boldsymbol{\lambda}_{s,lat}\cdot\boldsymbol{\phi}, \ \ \forall\boldsymbol{\phi}\in\mathbf{H}^1(\Omega_{\eta_1(t)}),\\
b(\mathbf{u},\psi) = 0, \ \ \forall\psi\in L^2(\Omega_{\eta_1(t)}),\\
\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma_i}\mathbf{g}^i\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma_i), \qquad \int_{\Gamma_0}\mathbf{u}\cdot\boldsymbol{\tau} = 0, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma_0),\\
\displaystyle\int_{\Gamma^+_{\eta_1(t)}}\mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^+_{\eta_1(t)}}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^+_{\eta_1(t)}),\\
\displaystyle\int_{\Gamma^-_{\eta_1(t)}}\mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^-_{\eta_1(t)}}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^-_{\eta_1(t)}),\\
\displaystyle\int_{\Gamma^\ell_{\eta_1(t)}}\mathbf{u}\cdot\boldsymbol{\tau} = \int_{\Gamma^\ell_{\eta_1(t)}}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}, \ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac12}(\Gamma^\ell_{\eta_1(t)}),\\
\displaystyle\int_0^{\ell_s}(\partial_t\eta_1)\zeta = \int_0^{\ell_s}\eta_2\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
\displaystyle\int_0^{\ell_s}(\partial_t\eta_2)\zeta = a^1_s(\eta_1,\zeta) + a^2_s(\eta_2,\zeta) - \int_{\Gamma^+_{\eta_1(t)}}\boldsymbol{\lambda}_{s,top}\cdot\vec{\mathbf{e}}_2\zeta - \int_{\Gamma^-_{\eta_1(t)}}\boldsymbol{\lambda}_{s,bot}\cdot\vec{\mathbf{e}}_2\zeta,\\
\displaystyle\qquad\qquad + \sum_{j=1}^{N_c}\int_0^{\ell_s} w_j(z_1)f_j(t)\zeta(z_1) + \int_0^{\ell_s} f_s\zeta, \ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),
\end{array}\right. \tag{6.1.8}$$

where the bilinear forms a_f, b, a_s^1 and a_s^2 are given by

$$\begin{aligned} a_f(\mathbf{v},\boldsymbol{\phi}) &= -2\nu \int_{\Omega_{\eta_1(t)}} \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\phi}), \quad b(\boldsymbol{\phi},q) = \int_{\Omega_{\eta_1(t)}} (\operatorname{div}\boldsymbol{\phi})\, q, \\ a_s^1(\eta_1,\zeta) &= -\alpha \int_0^{\ell_s} \Delta\eta_1 \cdot \Delta\zeta, \quad a_s^2(\eta_2,\zeta) = -\gamma \int_0^{\ell_s} \Delta\eta_2 \cdot \Delta\zeta, \end{aligned} \tag{6.1.9}$$

while the trilinear form c is defined by

$$c(\mathbf{u},\mathbf{v},\boldsymbol{\phi}) = -\int_{\Omega_{\eta_1(t)}} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\phi}. \tag{6.1.10}$$

System (6.1.8) has to be completed with initial conditions.

The remainder of this chapter is organized as follows. In Section 6.2, we present the main ideas of the stabilization of the semi-discrete system in space associated to (6.1.8). Next, in Section 6.3, we recall the time-marching process described in Section 4.3 of Chapter 4. Then, in Section 6.4, we present numerical experiments.

6.2 Stabilization of the semi-discrete system

The goal of this section is to present the main ideas about the stabilization of the semi-discrete system obtained by approximating by a finite element the system (6.1.8) rewritten in a reference configuration. In this section we consider the system without perturbation at the inflow and $f_s = 0$.
We first introduce the mapping $\mathcal{A}(t,\cdot) : \Omega \longrightarrow \Omega_{\eta_1(t)}$ defined by

$$\mathcal{A}(t,\cdot) = I + \mathbf{d}(t,\cdot), \tag{6.2.1}$$

where the displacement $\mathbf{d}(t,\cdot)$ is solution of the elliptic equation

$$\Delta\mathbf{d} = 0 \ \text{ in } \Omega, \quad \mathbf{d} = \eta_1\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{d} = 0 \ \text{ on } \Gamma\setminus\Gamma_s. \tag{6.2.2}$$

Let us notice that, since $\eta_{1,t} = \eta_2$ on Γ_s, the auxiliar variable $\mathbf{w}(t,\cdot) := \partial_t\mathcal{A}(t,\cdot)$ satisfies the elliptic equation

$$\Delta\mathbf{w} = 0 \ \text{ in } \Omega, \quad \mathbf{w} = \eta_2\vec{\mathbf{e}}_2 \ \text{ on } \Gamma_s, \quad \mathbf{w} = 0 \ \text{ on } \Gamma\setminus\Gamma_s. \tag{6.2.3}$$

In order to take into account the Dirichlet boundary conditions on Γ in systems (6.2.2) and (6.2.3), we introduce the Lagrange multipliers $\boldsymbol{\lambda}^d = (\boldsymbol{\lambda}_s^d, \boldsymbol{\lambda}_f^d)^\top$ and $\boldsymbol{\lambda}^w = (\boldsymbol{\lambda}_s^w, \boldsymbol{\lambda}_f^w)^\top$, where:

$$\begin{aligned} &\boldsymbol{\lambda}_s^d \text{ is the multiplier associated to the boundary data prescribed on } \Gamma_s, \\ &\boldsymbol{\lambda}_f^d \text{ is the multiplier associated to the null boundary data prescribed on } \Gamma\setminus\Gamma_s, \\ &\boldsymbol{\lambda}_s^w \text{ is the multiplier associated to the boundary data prescribed on } \Gamma_s, \\ &\boldsymbol{\lambda}_f^w \text{ is the multiplier associated to the null boundary data prescribed on } \Gamma\setminus\Gamma_s. \end{aligned} \tag{6.2.4}$$

Using the transformation $\mathcal{A}$ defined in (6.2.1), we obtain that the linearization of system (6.1.8) around the stationary solution $(\mathbf{u}, p, \boldsymbol{\lambda}, \eta_1, \eta_2, \mathbf{d}, \boldsymbol{\lambda}^d, \mathbf{w}, \boldsymbol{\lambda}^w) = (\mathbf{u}_s, p_s, \boldsymbol{\lambda}^s, 0, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, where the triple $(\mathbf{u}_s, p_s, \boldsymbol{\lambda}^s)$ is a stationary solution of system (6.1.1), with $\boldsymbol{\lambda}^s$ representing the Lagrange multiplier associated with the Dirichlet boundary condition, reads as follows:

Find $\mathbf{v} \in H^1_{loc}(0,\infty;\mathbf{H}^{-\frac{1}{2}+\alpha}(\Omega)) \cap L^2_{loc}(0,\infty;\mathbf{H}^1(\Omega))$, $p \in L^2_{loc}(0,\infty;L^2(\Omega))$, $\eta_1,\eta_2 \in L^2_{loc}(0,\infty;H^2_{\{0\}}(0,\ell_s))$, $\boldsymbol{\lambda}^u \in L^2_{loc}(0,\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma_d))$, $\mathbf{d},\mathbf{w} \in L^2_{loc}(0;\infty;\mathbf{H}^1(\Omega))$, $\boldsymbol{\lambda}^d,\boldsymbol{\lambda}^w \in L^2_{loc}(0,\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))$ such that

$$\left\{\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathbf{v}\cdot\boldsymbol{\phi} = \widetilde{a}_f(\mathbf{v},\boldsymbol{\phi}) + b(\boldsymbol{\phi},p) + \int_{\Gamma_i}\boldsymbol{\lambda}_i\cdot\boldsymbol{\phi} + \int_{\Gamma_0}\boldsymbol{\lambda}_0\cdot\boldsymbol{\phi} + \int_{\Gamma^+}\boldsymbol{\lambda}_{s,top}\cdot\boldsymbol{\phi} + \int_{\Gamma^-}\boldsymbol{\lambda}_{s,bot}\cdot\boldsymbol{\phi}\\
&\qquad\qquad + \int_{\Gamma^\ell}\boldsymbol{\lambda}_{s,lat}\cdot\boldsymbol{\phi} + \int_\Omega A_1\mathbf{w}\cdot\boldsymbol{\phi} + \int_\Omega A_2\mathbf{d}\cdot\boldsymbol{\phi},\ \ \forall\boldsymbol{\phi}\in\mathbf{H}^1(\Omega),\\
&b(\mathbf{v},\psi) = \int_\Omega A_3\mathbf{d}\psi,\ \ \forall\psi\in L^2(\Omega),\\
&\int_{\Gamma_i}\mathbf{v}\cdot\boldsymbol{\tau} = 0,\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_i),\\
&\int_{\Gamma_0}\mathbf{v}\cdot\boldsymbol{\tau} = 0,\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_0),\\
&\int_{\Gamma^+}\mathbf{v}\cdot\boldsymbol{\tau} = \int_{\Gamma^+}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^+),\\
&\int_{\Gamma^-}\mathbf{v}\cdot\boldsymbol{\tau} = \int_{\Gamma^-}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^-),\\
&\int_{\Gamma^\ell}\mathbf{v}\cdot\boldsymbol{\tau} = \int_{\Gamma^\ell}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau},\ \ \forall\boldsymbol{\tau}\in\mathbf{H}^{-\frac{1}{2}}(\Gamma^\ell),\\
&\frac{d}{dt}\int_0^{\ell_s}\eta_1\zeta = \int_0^{\ell_s}\eta_2\zeta,\ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
&\frac{d}{dt}\int_0^{\ell_s}\eta_2\zeta = a_s^1(\eta_1,\zeta) + a_s^2(\eta_2,\zeta) - \int_{\Gamma^+}\boldsymbol{\lambda}_{s,top}\cdot\vec{\mathbf{e}}_2\zeta - \int_{\Gamma^-}\boldsymbol{\lambda}_{s,bot}\cdot\vec{\mathbf{e}}_2\zeta + \int_0^{\ell_s}A_4\mathbf{d}\zeta\\
&\qquad\qquad + \sum_{j=1}^{N_c}\int_0^{\ell_s}w_j(z_1)f_j(t)\zeta(z_1),\ \ \forall\zeta\in H^2_{\{0\}}(0,\ell_s),\\
&\int_\Omega\nabla\mathbf{d}:\nabla\boldsymbol{\varphi} - \int_{\Gamma_s}\boldsymbol{\lambda}_s^d\cdot\boldsymbol{\varphi} - \int_{\Gamma\setminus\Gamma_s}\boldsymbol{\lambda}_f^d\cdot\boldsymbol{\varphi} = 0,\ \ \text{for all }\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega),\\
&\int_{\Gamma_s}\mathbf{d}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s}\eta_1\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0,\quad\text{for all }\boldsymbol{\tau}_s\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_s),\\
&\int_{\Gamma\setminus\Gamma_s}\mathbf{d}\cdot\boldsymbol{\tau}_f = 0,\ \ \text{for all }\boldsymbol{\tau}_f\in\mathbf{H}^{-\frac{1}{2}}(\Gamma\setminus\Gamma_s),\\
&\int_\Omega\nabla\mathbf{w}:\nabla\boldsymbol{\varphi} - \int_{\Gamma_s}\boldsymbol{\lambda}_s^w\cdot\boldsymbol{\varphi} - \int_{\Gamma\setminus\Gamma_s}\boldsymbol{\lambda}_f^w\cdot\boldsymbol{\varphi} = 0,\ \ \text{for all }\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega),\\
&\int_{\Gamma_s}\mathbf{w}\cdot\boldsymbol{\tau}_s - \int_{\Gamma_s}\eta_2\vec{\mathbf{e}}_2\cdot\boldsymbol{\tau}_s = 0,\quad\text{for all }\boldsymbol{\tau}_s\in\mathbf{H}^{-\frac{1}{2}}(\Gamma_s),\\
&\int_{\Gamma\setminus\Gamma_s}\mathbf{w}\cdot\boldsymbol{\tau}_f = 0,\ \ \text{for all }\boldsymbol{\tau}_f\in\mathbf{H}^{-\frac{1}{2}}(\Gamma\setminus\Gamma_s),
\end{aligned}\right.\tag{6.2.5}$$

where

$$\widetilde{a}_f(\mathbf{v},\boldsymbol{\phi}) = -2\nu\int_\Omega\varepsilon(\mathbf{v}):\varepsilon(\boldsymbol{\phi}) - \int_\Omega((\mathbf{u}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_s)\cdot\boldsymbol{\phi},$$

while the bilinear forms b, a_s^1 and a_s^2 are defined in (6.1.9). System (6.2.5) has to be completed with initial conditions. The linear operators A_1, A_2, A_3 and A_4 are determined (in their discretized form) using a *GetFEM++ library* [RP] routine.

6.2.1 Semi-discrete approximation

We introduce the finite-dimensional spaces $\mathbf{V}_h \subset \mathbf{H}^1(\Omega)$ for the velocity, $P_h \subset L^2(\Omega)$ for the pressure, $\mathbf{D}_h^v \subset \mathbf{L}^2(\Gamma_d)$ and $\mathbf{D}_h^{dw} \subset \mathbf{L}^2(\Gamma)$ for the multipliers associated to $\mathbf{v}$ and $(\mathbf{d},\mathbf{w})$, respectively, and $S_h \subset H^2_{\{0\}}(0,\ell_s)$ for the structure's displacement and its velocity. We denote by $(\boldsymbol{\phi}_i)_{1\le i\le N_v}$ a basis of $\mathbf{V}_h$, $(q_i)_{1\le i\le N_p}$ a basis of P_h, $(\boldsymbol{\mu}_i^v)_{1\le i\le N_\lambda^v}$ a basis of $\mathbf{D}_h^v$, $(\boldsymbol{\mu}_i^{dw})_{1\le i\le N_\lambda^{dw}}$ a basis of $\mathbf{D}_h^{dw}$ and $(\zeta_i)_{1\le i\le N_s}$ a basis of S_h. We then set

$$
\begin{aligned}
&\mathbf{v} = \sum_{i=1}^{N_v} v_i \boldsymbol{\phi}_i, \ \ \mathbf{u}^0 = \sum_{i=1}^{N_v} u_i^0 \boldsymbol{\phi}_i, \ \ p = \sum_{i=1}^{N_p} p_i q_i, \ \ \eta_1 = \sum_{i=1}^{N_s} \eta_1^i \zeta_i, \ \ \eta_2 = \sum_{i=1}^{N_s} \eta_2^i \zeta_i,\\
&\eta_1^0 = \sum_{i=1}^{N_s} \eta_{1,0}^i \zeta_i, \ \ \eta_2^0 = \sum_{i=1}^{N_s} \eta_{2,0} \zeta_i, \ \ \boldsymbol{\lambda}_f^v = \sum_{i=1}^{N_\lambda^v} \lambda_{f,i}^v \boldsymbol{\mu}_i^v, \ \ \boldsymbol{\lambda}_s^v = \sum_{i=1}^{N_\lambda^v} \lambda_{s,i}^v \boldsymbol{\mu}_i^v,\\
&\mathbf{d} = \sum_{i=1}^{N_v} d_i \boldsymbol{\phi}_i, \ \ \boldsymbol{\lambda}_f^d = \sum_{i=1}^{N_\lambda^{dw}} \lambda_{f,i}^d \boldsymbol{\mu}_i^{dw}, \ \ \boldsymbol{\lambda}_s^d = \sum_{i=1}^{N_\lambda^{dw}} \lambda_{s,i}^d \boldsymbol{\mu}_i^{dw},\\
&\mathbf{w} = \sum_{i=1}^{N_v} w_i \boldsymbol{\phi}_i, \ \ \boldsymbol{\lambda}_f^w = \sum_{i=1}^{N_\lambda^{dw}} \lambda_{f,i}^w \boldsymbol{\mu}_i^{dw}, \ \ \boldsymbol{\lambda}_s^w = \sum_{i=1}^{N_\lambda^{dw}} \lambda_{s,i}^w \boldsymbol{\mu}_i^{dw}.
\end{aligned}
$$

We also introduce the corresponding coordinate vectors,

$$
\begin{aligned}
&\mathbf{V} = (v_1, \cdots, v_{N_v})^\top, \ \ \mathbf{V}^0 = (u_1^0, \cdots, u_{N_v}^0)^\top, \ \ \mathbf{P} = (p_1, \cdots, p_{N_p})^\top, \ \ \mathbf{N}_1 = (\eta_1^1, \cdots, \eta_1^{N_s})^\top,\\
&\mathbf{N}_2 = (\eta_2^1, \cdots, \eta_2^{N_s})^\top, \ \ \mathbf{N}_1^0 = (\eta_{1,0}^1, \cdots, \eta_{1,0}^{N_s})^\top, \ \ \mathbf{N}_2^0 = (\eta_{2,0}^1, \cdots, \eta_{2,0}^{N_s})^\top,\\
&\boldsymbol{\Lambda}_f^v = (\lambda_{f,1}^v, \cdots, \lambda_{f,N_\lambda^v}^v)^\top, \ \ \boldsymbol{\Lambda}_s^v = (\lambda_{s,1}^v, \cdots, \lambda_{s,N_\lambda^v}^v)^\top, \ \ \boldsymbol{\Lambda}_f^d = (\lambda_{f,1}^d, \cdots, \lambda_{f,N_\lambda^{dw}}^d)^\top,\\
&\boldsymbol{\Lambda}_s^d = (\lambda_{s,1}^d, \cdots, \lambda_{s,N_\lambda^{dw}}^d)^\top, \ \ \boldsymbol{\Lambda}_f^w = (\lambda_{f,1}^w, \cdots, \lambda_{f,N_\lambda^{dw}}^w)^\top, \ \ \boldsymbol{\Lambda}_s^w = (\lambda_{s,1}^w, \cdots, \lambda_{s,N_\lambda^{dw}}^w)^\top.
\end{aligned}
$$

For all $1 \le i, j \le N_v$, $1 \le k \le N_p$, $1 \le \ell, m \le N_\lambda^v$, $1 \le r, t \le N_s$, $1 \le o \le N_\lambda^{dw}$, we introduce the matrices

$$
\begin{aligned}
&(A_{vv})_{ij} = \widetilde{a}_f(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j), \ \ (A_{vp})_{ik} = b(\boldsymbol{\phi}_i, p_k), \ \ (A_{v\lambda_f^v})_{i\ell} = \int_{\Gamma_d \setminus \Gamma_s} \boldsymbol{\mu}_\ell^v \cdot \boldsymbol{\phi}_i, \ \ (A_{v\lambda_s^v})_{i\ell} = \int_{\Gamma_s} \boldsymbol{\mu}_\ell^v \cdot \boldsymbol{\phi}_i,\\
&(M_{vv})_{ij} = \int_\Omega \boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_i, \ \ (\widetilde{A}_{\eta_2 \lambda_s^v})_{ro} = -\int_{\Gamma_s^+ \cup \Gamma_s^-} \boldsymbol{\mu}_o^v \cdot \vec{\mathbf{e}}_2 \zeta_r, \ \ (A_{\lambda_s^v \eta_2})_{or} = -\int_{\Gamma_s} \boldsymbol{\mu}_o^v \cdot \vec{\mathbf{e}}_2 \zeta_r,\\
&(A_{vw})_{ij} = \int_\Omega A_1 \boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_i, \ \ (A_{vd})_{ij} = \int_\Omega A_2 \boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_i, \ \ (A_{dp})_{ik} = -\int_\Omega A_3 \boldsymbol{\phi}_i \, p_k,\\
&(M_{\lambda^v \lambda^v})_{m\ell} = \int_{\Gamma_d} \boldsymbol{\mu}_\ell^v \cdot \boldsymbol{\mu}_m^v, \ \ (A_{\eta_1 \eta_2})_{rt} = (M_{\eta\eta})_{rt} = \int_0^{\ell_s} \zeta_r \cdot \zeta_t, \ \ (A_{\eta_2 \eta_1})_{rt} = -\alpha \int_0^{\ell_s} \Delta\zeta_r \cdot \Delta\zeta_t,\\
&(A_{\eta_2 \eta_2})_{rt} = -\gamma \int_0^{\ell_s} \Delta\zeta_r \cdot \Delta\zeta_t, \ \ (A_{\eta_2 d})_{ir} = \int_0^{\ell_s} A_4 \boldsymbol{\phi}_i \zeta_r,\\
&(A_{dd})_{ij} = (A_{ww})_{ij} = \int_\Omega \nabla\boldsymbol{\phi}_i : \nabla\boldsymbol{\phi}_j, \ \ (A_{d\lambda_f^d})_{io} = (A_{w\lambda_f^w})_{io} = -\int_{\Gamma \setminus \Gamma_s} \boldsymbol{\phi}_i \cdot \boldsymbol{\mu}_o^{dw},\\
&(A_{d\lambda_s^d})_{io} = (A_{w\lambda_s^w})_{io} = -\int_{\Gamma_s} \boldsymbol{\phi}_i \cdot \boldsymbol{\mu}_o^{dw}, \ \ (A_{\lambda_s^d \eta_1})_{or} = (A_{\lambda_s^w \eta_2})_{or} = \int_{\Gamma_s} \boldsymbol{\mu}_o^{dw} \cdot \vec{\mathbf{e}}_2 \zeta_r.
\end{aligned}
$$

We also set

$$
N_z = N_v + 2N_s, \ \ N_\theta = N_p + N_{\lambda^v}, \ \ N_{\overline{w}} = N_{\overline{d}} = N_v + N_{\lambda^{dw}} \ \text{ and } \ N = N_z + N_\theta + N_{\overline{w}} + N_{\overline{d}},
$$

and define the vectors

$$
\begin{aligned}
&\mathbf{Z} = (\mathbf{V} \ \ \mathbf{N}_1 \ \ \mathbf{N}_2)^\top, \ \ \boldsymbol{\Lambda}^v = (\boldsymbol{\Lambda}_f^v \ \ \boldsymbol{\Lambda}_s^v)^\top, \ \ \boldsymbol{\Theta} = (\mathbf{P} \ \ \boldsymbol{\Lambda}^v)^\top,\\
&\mathbf{W} = (w_1, \cdots, w_{N_v})^\top, \ \ \boldsymbol{\Lambda}^w = (\boldsymbol{\Lambda}_f^w \ \ \boldsymbol{\Lambda}_s^w)^\top, \ \ \overline{\mathbf{W}} = (\mathbf{W} \ \ \boldsymbol{\Lambda}^w)^\top,\\
&\mathbf{D} = (d_1, \cdots, d_{N_v})^\top, \ \ \boldsymbol{\Lambda}^d = (\boldsymbol{\Lambda}_f^d \ \ \boldsymbol{\Lambda}_s^d)^\top, \ \ \overline{\mathbf{D}} = (\mathbf{D} \ \ \boldsymbol{\Lambda}^d)^\top, \ \ \mathbf{F} = (f_1, \cdots, f_{N_c})^\top.
\end{aligned}
$$

We now introduce the mass matrix $M \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$, the stiffness matrix $A \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$, and the control operator $B \in \mathcal{L}(\mathbb{R}^{N_c}, \mathbb{R}^N)$:

$$M = \begin{pmatrix} M_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{zz} & A_{z\theta} & A_{z\overline{w}} & A_{z\overline{d}} \\ A_{\theta z} & 0 & 0 & A_{\theta\overline{d}} \\ A_{\overline{w}z} & 0 & A_{\overline{ww}} & 0 \\ A_{\overline{d}z} & 0 & 0 & A_{\overline{dd}} \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 0 \\ M_{\eta\eta}\mathbf{L} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M_{zz} = \begin{pmatrix} M_{vv} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & 0 & M_{\eta\eta} \end{pmatrix}, \quad A_{zz} = \begin{pmatrix} A_{vv} & 0 & 0 \\ 0 & 0 & A_{\eta_1\eta_2} \\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2} \end{pmatrix} \quad A_{z\theta} = \begin{pmatrix} A_{vp} & A_{v\lambda_f^v} & A_{v\lambda_s^v} \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{\eta_2\lambda_s^v} \end{pmatrix},$$

$$A_{z\overline{w}} = \begin{pmatrix} A_{vw} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{z\overline{d}} = \begin{pmatrix} A_{vd} & 0 & 0 \\ 0 & 0 & 0 \\ A_{\eta_2 d} & 0 & 0 \end{pmatrix}, \quad A_{\theta z} = \begin{pmatrix} A_{vp}^\top & 0 & 0 \\ A_{v\lambda_f^v}^\top & 0 & 0 \\ A_{v\lambda_s^v}^\top & 0 & A_{\lambda_s^v\eta_2} \end{pmatrix},$$

$$A_{\theta\overline{d}} = \begin{pmatrix} A_{pd} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\overline{w}z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{\lambda_s^w\eta_2} \end{pmatrix}, \quad A_{\overline{ww}} = \begin{pmatrix} A_{ww} & A_{w\lambda_f^w} & A_{w\lambda_s^w} \\ A_{w\lambda_f^w}^\top & 0 & 0 \\ A_{w\lambda_s^w}^\top & 0 & 0 \end{pmatrix},$$

$$A_{\overline{d}z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{\lambda_s^d\eta_1} & 0 \end{pmatrix}, \quad A_{\overline{dd}} = \begin{pmatrix} A_{dd} & A_{w\lambda_f^d} & A_{d\lambda_s^d} \\ A_{d\lambda_f^d}^\top & 0 & 0 \\ A_{d\lambda_s^d}^\top & 0 & 0 \end{pmatrix},$$

$$\mathbf{L} = (\mathbf{L}_1 \cdots \mathbf{L}_{N_c}), \quad \mathbf{L}_j = (w_j^1, \cdots, w_j^{N_s})^\top, \quad j = 1, \ldots, N_c.$$

Thus, the spatial semi-discretization of system (6.2.5) can be written as

$$M \frac{d}{dt} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Theta} \\ \overline{\boldsymbol{W}} \\ \overline{\boldsymbol{D}} \end{pmatrix} = A \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Theta} \\ \overline{\boldsymbol{W}} \\ \overline{\boldsymbol{D}} \end{pmatrix} + B\mathbf{F}. \tag{6.2.6}$$

6.2.2 Eliminating the fluid domain variables $\overline{\mathbf{W}}$ and $\overline{\mathbf{D}}$

In order to eliminate the fluid domain variables $\overline{\mathbf{W}}$ and $\overline{\mathbf{D}}$, we follow the same idea used in [FAC].

From the third and fourth equation in (6.2.6) we deduce that

$$\overline{\mathbf{W}} = -A_{\overline{ww}}^{-1} A_{\overline{w}z} \mathbf{Z} \quad \text{and} \quad \overline{\mathbf{D}} = -A_{\overline{dd}}^{-1} A_{\overline{d}z} \mathbf{Z}. \tag{6.2.7}$$

We recall tha the matrices $A_{\overline{ww}}$ and $A_{\overline{dd}}$ are given by

$$A_{\overline{ww}} = \begin{pmatrix} A_{ww} & A_{w\lambda_f^w} & A_{w\lambda_s^w} \\ A_{w\lambda_f^w}^\top & 0 & 0 \\ A_{w\lambda_s^w}^\top & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{\overline{dd}} = \begin{pmatrix} A_{dd} & A_{w\lambda_f^d} & A_{d\lambda_s^d} \\ A_{d\lambda_f^d}^\top & 0 & 0 \\ A_{d\lambda_s^d}^\top & 0 & 0 \end{pmatrix}.$$

The invertibility of these matrices holds provided that the inf-sup condition is satisfied by the finite element chosen to approximate the variables $(\mathbf{w}, \boldsymbol{\lambda}^w)$ (respectively, $(\mathbf{d}, \boldsymbol{\lambda}^d)$). Using the expressions of $\overline{\mathbf{W}}$ and $\overline{\mathbf{D}}$ given in (6.2.7) in (6.2.6), we obtain the following reduced system:

$$\widetilde{M}\frac{d}{dt}\begin{pmatrix}\mathbf{Z}\\ \mathbf{\Theta}\end{pmatrix} = \widetilde{A}\begin{pmatrix}\mathbf{Z}\\ \mathbf{\Theta}\end{pmatrix} + \widetilde{B}\mathbf{F}, \tag{6.2.8}$$

where the matrices $\widetilde{M}, \widetilde{A} \in \mathcal{L}(\mathbb{R}^{(N_z+N_\theta)}, \mathbb{R}^{(N_z+N_\theta)})$ and the operator $\widetilde{B} \in \mathcal{L}(\mathbb{R}^{N_c}, \mathbb{R}^{N_z+N_\theta})$ are given by

$$\widetilde{M} = \begin{pmatrix} M_{zz} & 0\\ 0 & 0\end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} \widetilde{A}_{zz} & A_{z\theta}\\ \widetilde{A}_{\theta z} & 0\end{pmatrix}, \quad \text{and} \quad \widetilde{B} = \begin{pmatrix} 0\\ 0\\ M_{\eta\eta}\mathbf{L}\\ 0\end{pmatrix}, \tag{6.2.9}$$

with

$$\widetilde{A}_{zz} = A_{zz} - A_{z\overline{w}}A_{\overline{ww}}^{-1}A_{\overline{w}z} - A_{z\overline{d}}A_{\overline{dd}}^{-1}A_{\overline{d}z} \quad \text{and} \quad \widetilde{A}_{\theta z} = A_{\theta z} - A_{\theta\overline{d}}A_{\overline{dd}}^{-1}A_{\overline{d}z}.$$

We remark that Subsections 6.2.3, 6.2.4 and 6.2.5 collect some of the results presented in [Del18, Chapter 3] and [Ndi16].

6.2.3 Finite dimensional controlled system

The aim of this subsection is to reformulate system (6.2.8) solely in terms of the variable $\mathbf{Z}$ by eliminating the multiplier $\mathbf{\Theta}$, and to express $\mathbf{\Theta}$ as a function of $\mathbf{Z}$ and the given data. In order to do that, we will introduce the projection Π onto $\mathrm{Ker}(\widetilde{A}_{\theta z})$ parallel to $\mathrm{Im}(M_{zz}^{-1}A_{z\theta})$. The characterization of such a projector, as presented in [Del18, Lemma 3.3.2, p.110], is given in the following proposition.

Proposition 6.2.1. *The projector Π of $\mathbb{R}^{N_z}$ onto $\mathrm{Ker}(\widetilde{A}_{\theta z})$ parallel to $\mathrm{Im}(M_{zz}^{-1}A_{z\theta})$ is defined by*

$$\Pi = I - M_{zz}^{-1}A_{z\theta}\left(\widetilde{A}_{\theta z}M_{zz}^{-1}A_{z\theta}\right)^{-1}\widetilde{A}_{\theta z}.$$

The projector $\Pi^\top$ of $\mathbb{R}^{N_z}$ onto $\mathrm{Ker}(\widetilde{A}_{z\theta}^\top M_{zz}^{-1})$ parallel to $\mathrm{Im}(\widetilde{A}_{\theta z}^\top)$ is defined by

$$\Pi^\top = I - \widetilde{A}_{\theta z}^\top\left(A_{z\theta}^\top M_{zz}^{-1}\widetilde{A}_{\theta z}^\top\right)^{-1}A_{z\theta}^\top M_{zz}^{-1}.$$

Then, using this proposition, we deduce the following result:

Proposition 6.2.2. *A pair $(\mathbf{Z}, \mathbf{\Theta})$ is solution of (6.2.8) if and only if $(\mathbf{Z}, \mathbf{\Theta})$ is solution to the system*

$$\begin{cases} \dfrac{d}{dt}\Pi\mathbf{Z} = \mathbb{A}\Pi\mathbf{Z} + \mathbb{B}\mathbf{F},\\ (I-\Pi)\mathbf{Z} = 0,\\ \mathbf{\Theta} = -(\widetilde{A}_{\theta z}M_{zz}^{-1}A_{z\theta})^{-1}\widetilde{A}_{\theta z}M_{zz}^{-1}\widetilde{A}_{zz}\mathbf{Z} - (\widetilde{A}_{\theta z}M_{zz}^{-1}A_{z\theta})^{-1}\widetilde{A}_{\theta z}M_{zz}^{-1}\widetilde{B}\mathbf{F}, \end{cases} \tag{6.2.10}$$

where $\mathbb{A} = \Pi M_{zz}^{-1}\widetilde{A}_{zz}$ and $\mathbb{B} = \Pi M_{zz}^{-1}\widetilde{B}$.

6.2.4 Spectral decomposition of the operators

In this subsection, we are interested in finding a decomposition of $\mathrm{Ker}(\widetilde{A}_{\theta z})$ into a sum of generalized eigenspaces of the operator $\mathbb{A}$. We consider the eigenvalue problem

$$\mu \in \mathbb{C}^*, \quad \mathbf{Z} \in \mathrm{Ker}(\widetilde{A}_{\theta z}), \quad \mathbf{Z} \neq \mathbf{0}, \quad \mathbb{A}\mathbf{Z} = \mu\mathbf{Z}. \tag{6.2.11}$$

We define the operator $\mathbb{A}^\# = \widetilde{\Pi}M_{zz}^{-1}\widetilde{A}_{zz}^\top$, where $\widetilde{\Pi} = I - M_{zz}^{-1}\widetilde{A}_{\theta z}^\top(A_{z\theta}^\top M_{zz}^{-1}\widetilde{A}_{\theta z}^\top)^{-1}A_{z\theta}^\top$. We now consider the eigenvalue problem for $\mathbb{A}^\#$

$$\mu \in \mathbb{C}^*, \quad \mathbf{\Phi} \in \mathrm{Ker}(A_{z\theta}^\top), \quad \mathbf{\Phi} \neq \mathbf{0}, \quad \mathbb{A}^\#\mathbf{\Phi} = \mu\mathbf{\Phi}. \tag{6.2.12}$$

We recall that a vector $\mathbf{Z}_k \in \mathbb{C}^{N_z} \setminus \{\mathbf{0}\}$ is a *generalized eigenvector* for problem (6.2.11) associated with a solution $(\mu, \mathbf{Z})$ of (6.2.11) when

$$\mathbf{Z}_k \in \mathrm{Ker}(\widetilde{A}_{\theta z}), \ \ \mathbf{Z}_k \neq \mathbf{0}, \ \ (\mathbb{A} - \mu)^k \mathbf{Z}_k = \mathbf{Z} \ \text{ for some } k \in \mathbb{N}. \tag{6.2.13}$$

Similarly, we have that the vector $\boldsymbol{\Phi}_k \in \mathbb{C}^{N_z} \setminus \{\mathbf{0}\}$ is a *generalized eigenvector* for problem (6.2.12) associated with a solution $(\mu, \boldsymbol{\Phi})$ of (6.2.12) when

$$\boldsymbol{\Phi}_k \in \mathrm{Ker}(A_{z\theta}^\top), \ \ \boldsymbol{\Phi}_k \neq \mathbf{0}, \ \ (\mathbb{A}^\# - \mu)^k \boldsymbol{\Phi}_k = \boldsymbol{\Phi} \ \text{ for some } k \in \mathbb{N}. \tag{6.2.14}$$

In the following proposition we establish the equivalence between the eigenvalue problems associated to $\mathbb{A}$ (see problem (6.2.11)) and the one associated to the pair $(\widetilde{M}, \widetilde{A})$ (to be defined in the statement of the proposition, see problem (6.2.15)).

Proposition 6.2.3. *A pair $(\mu, \mathbf{Z}) \in \mathbb{C}^* \times \mathbb{C}^{N_z}$ is a solution to the eigenvalue problem (6.2.11) if and only if $(\mu, \mathbf{Z}, \boldsymbol{\Theta}_z)$, with $\boldsymbol{\Theta}_z = -(\widetilde{A}_{\theta z} M_{zz}^{-1} A_{z\theta})^{-1} \widetilde{A}_{\theta z} M_{zz}^{-1} A_{zz} \mathbf{Z}$, is a solution of the eigenvalue problem*

$$\mu \in \mathbb{C}^*, \ \ \mathbf{Z} \in \mathbb{C}^{N_z} \setminus \{\mathbf{0}\}, \ \ \boldsymbol{\Theta}_z \in \mathbb{C}^{N_\theta}, \ \ \widetilde{A} \begin{pmatrix} \mathbf{Z} \\ \boldsymbol{\Theta} \end{pmatrix} = \mu \widetilde{M} \begin{pmatrix} \mathbf{Z} \\ \boldsymbol{\Theta} \end{pmatrix}, \tag{6.2.15}$$

where the matrices $\widetilde{A}$ and $\widetilde{M}$ are defined in (6.2.9). In a similar way, the pair $(\mu, \boldsymbol{\Phi}) \in \mathbb{C}^* \times \mathbb{C}^{N_z}$ is a solution to the eigenvalue problem (6.2.12) if and only if $(\mu, \boldsymbol{\Phi}, \boldsymbol{\Theta}_\Phi)$, with $\boldsymbol{\Theta}_\Phi = -(A_{z\theta}^\top M_{zz}^{-1} \widetilde{A}_{\theta z}^\top)^{-1} A_{z\theta}^\top M_{zz}^{-1} A_{zz}^\top \boldsymbol{\Phi}$, is a solution of the eigenvalue problem

$$\mu \in \mathbb{C}^*, \ \ \boldsymbol{\Phi} \in \mathbb{C}^{N_z} \setminus \{\mathbf{0}\}, \ \ \boldsymbol{\Theta}_\Phi \in \mathbb{C}^{N_\theta}, \ \ \widetilde{A}^\top \begin{pmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Theta}_\Phi \end{pmatrix} = \mu \widetilde{M} \begin{pmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Theta}_\Phi \end{pmatrix}. \tag{6.2.16}$$

The proof of the preceding result is provided in [Del18, Lemmas 3.41 and 3.42, p.118-119]. A similar statement can be established for the generalized eigenvalue problems (see [Del18, Theorem 3.4.3, p.119]).

In what follows, we will denote by N_π the dimension of the subspace $\mathrm{Ker}(\widetilde{A}_{\theta z})$.

Once the equivalence of the eigenvalue problems is established, an important point in the analysis is the existence of appropriate bases for $\mathrm{Ker}(\widetilde{A}_{\theta z})$ and $\mathrm{Ker}(A_{z\theta}^\top)$ that satisfy a biorthogonality condition. These bases will then be used to characterize suitable projector operators (see (6.2.19)). The existence of such bases is established in the following two theorems, whose proofs can be found in [Del18, Theorems 3.3.12, 3.3.14].

Theorem 6.2.1. *There exist two matrices $\Psi \in \mathcal{L}(\mathbb{C}^{N_z}, \mathbb{C}^{N_\pi})$ and $\widetilde{\Psi} \in \mathcal{L}(\mathbb{C}^{N_z}, \mathbb{C}^{N_\pi})$ satisfying the following assertions:*

(i) The columns of Ψ are eigenvectors and generalized eigenvectors of $\mathbb{A}$ and form a basis of $\mathrm{Ker}(\widetilde{A}_{\theta z})$.

(ii) The columns of $\widetilde{\Psi}$ are eigenvectors or generalized eigenvectors of $\mathbb{A}^\#$ and form a basis of $\mathrm{Ker}(A_{z\theta}^\top)$.

(iii) The following decompositions hold:

$$\Xi_{\mathbb{C}} = \widetilde{\Psi}^\top M_{zz} \mathbb{A} \Psi \in \mathcal{L}(\mathbb{C}^{N_\pi}, \mathbb{C}^{N_\pi}) \ \ \text{and} \ \ \Xi_{\mathbb{C}}^\top = \Psi^\top M_{zz} \mathbb{A}^\# \widetilde{\Psi} \in \mathcal{L}(\mathbb{C}^{N_\pi}, \mathbb{C}^{N_\pi}),$$

where $\Xi_{\mathbb{C}}$ is a decomposition of $\mathbb{A}$ into complex Jordan blocks.

(iv) The following biorthogonality condition holds:

$$\Psi^\top M_{zz} \widetilde{\Psi} = I_{N_\pi}. \tag{6.2.17}$$

Theorem 6.2.2. *There exist two matrices* $E \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_\pi})$ *and* $\widetilde{E} \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_\pi})$ *satisfying the following assertions:*

(*i*) *The columns of* E *form a basis of* $\mathrm{Ker}(\widetilde{A}_{\theta z})$, *while the columns of* $\widetilde{E}$ *form a basis of* $\mathrm{Ker}(A_{z\theta}^\top)$.

(*ii*) *The following decomposition holds:*

$$\Xi_{\mathbb{R}} = \widetilde{E}^\top M_{zz} \mathbb{A} E \in \mathcal{L}(\mathbb{R}^{N_\pi}, \mathbb{R}^{N_\pi}) \ \textit{and} \ \Xi_{\mathbb{R}}^\top = E^\top M_{zz} \mathbb{A}^\# \widetilde{E} \in \mathcal{L}(\mathbb{R}^{N_\pi}, \mathbb{R}^{N_\pi}),$$

where $\Xi_{\mathbb{R}}$ *is a real Jordan matrix.*

(*iv*) *The following biorthogonality condition holds:*

$$E^\top M_{zz} \widetilde{E} = I_{N_\pi}. \tag{6.2.18}$$

6.2.5 The projected dynamical system and computation of the feedback

Let $(\mu_j)_{1 \le j \le N_\pi}$ be the eigenvalues of the matrix $\mathbb{A}$. We denote by $G_{\mathbb{R}}(\mu_j)$ the real generalized eigenspace of $\mathbb{A}$ and by $G_{\mathbb{R}}^*(\mu_j)$ the real generalized eigenspace of $\mathbb{A}^\#$ associated to the eigenvalue μ_j. Given $\omega > 0$, we define the subset J_u of $\mathbb{N}^*$ by $J_u := \{j \in \mathbb{N}^* \mid \Re\lambda_j \ge -\omega\}$. We then introduce the spaces

$$\mathbb{Z}_u = \bigoplus_{j \in J_u} G_{\mathbb{R}}(\mu_j) \ \text{ and } \ \mathbb{Z}_u^* = \bigoplus_{j \in J_u} G_{\mathbb{R}}^*(\mu_j).$$

We denote by $\mathbf{E}^k$ (respectively, $\widetilde{\mathbf{E}}^k$) the k-th column of the matrix E (respectively, $\widetilde{E}$) presented in Theorem 6.2.2. We assume that

$$\mathbb{Z}_u = \mathrm{Vect}((\mathbf{E}^i)_{1 \le i \le N_u}) \ \text{ and } \ \mathbb{Z}_u^* = \mathrm{Vect}((\widetilde{\mathbf{E}}^i)_{1 \le i \le N_u}),$$

where $N_u = \dim(\mathbb{Z}_u)$. We also define

$$\mathbb{Z}_s = \mathrm{Vect}((\mathbf{E}^i)_{N_u+1 \le i \le N_\pi}) \ \text{ and } \ \mathbb{Z}_s^* = \mathrm{Vect}((\widetilde{\mathbf{E}}^i)_{N_u+1 \le i \le N_\pi}),$$

and $N_s = \dim(\mathbb{Z}_s)$. Then, we have

$$\mathrm{Ker}(\widetilde{A}_{\theta z}) = \mathbb{Z}_u \bigoplus \mathbb{Z}_s \ \text{ and } \ \mathrm{Ker}(A_{z\theta}^\top) = \mathbb{Z}_u^* \bigoplus \mathbb{Z}_s^*.$$

We define the following matrices:

- $E_u \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_u})$ the matrix whose columns are $(\mathbf{E}^i)_{1 \le i \le N_u}$,
- $E_s \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_s})$ the matrix whose columns are $(\mathbf{E}^i)_{N_u+1 \le i \le N_\pi}$,
- $\widetilde{E}_u \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_u})$ the matrix whose columns are $(\widetilde{\mathbf{E}}^i)_{1 \le i \le N_u}$,
- $\widetilde{E}_s \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{R}^{N_s})$ the matrix whose columns are $(\widetilde{\mathbf{E}}^i)_{N_u+1 \le i \le N_\pi}$,
- $\Xi_u \in \mathcal{L}(\mathbb{R}^{N_u}, \mathbb{R}^{N_u})$ and $\Xi_s \in \mathcal{L}(\mathbb{R}^{N_s}, \mathbb{R}^{N_s})$ are the matrices such that

$$\Xi_{\mathbb{R}} = \begin{pmatrix} \Xi_u & \mathbf{0} \\ \mathbf{0} & \Xi_s \end{pmatrix}.$$

We now introduce the projector $\Pi_u \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{Z}_u)$ (respectively, $\Pi_s \in \mathcal{L}(\mathbb{R}^{N_z}, \mathbb{Z}_s)$) of $\mathbb{R}^{N_z}$ onto $\mathbb{Z}_u$ (respectively, $\mathbb{Z}_s$) parallel to $\mathbb{Z}_s \oplus \mathrm{Ker}(\Pi)$ (respectively, $\mathbb{Z}_u \oplus \mathrm{Ker}(\Pi)$), which according to [Del18, Lemma 3.3.16, p.116] are characterized by

$$\Pi_u = E_u \widetilde{E}_u^\top M_{zz} \ \text{ and } \ \Pi_s = E_s \widetilde{E}_s^\top M_{zz}. \tag{6.2.19}$$

Proposition 6.2.4. *If* $\mathbf{Z}$ *is solution of system*

$$\begin{cases} \dfrac{d}{dt}\mathbf{Z}(t) = \mathbb{A}\mathbf{Z}(t) + \mathbb{B}\mathbf{F}(t), \quad t > 0, \\ \mathbf{Z}(0) = \mathbf{Z}_0, \end{cases} \tag{6.2.20}$$

then, $\zeta_u = \Pi_u \mathbf{Z}$ *and* $\zeta_s = \Pi_s \mathbf{Z}$ *obey the systems*

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{\zeta}_u(t) = \mathbb{A}_u\boldsymbol{\zeta}_u(t) + \mathbb{B}_u\mathbf{F}(t), \quad t > 0, \\ \boldsymbol{\zeta}_u(0) = \Pi_u\mathbf{Z}_0, \end{cases} \tag{6.2.21}$$

and

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{\zeta}_s(t) = \mathbb{A}_s\boldsymbol{\zeta}_s(t) + \mathbb{B}_s\mathbf{F}(t), \quad t > 0, \\ \boldsymbol{\zeta}_s(0) = \Pi_u\mathbf{Z}_0, \end{cases} \tag{6.2.22}$$

where $\mathbb{A}_u = \Pi_u M_{zz}^{-1}\widetilde{A}_{zz}$*,* $\mathbb{A}_s = \Pi_s M_{zz}^{-1}\widetilde{A}_{zz}$*,* $\mathbb{B}_u = \Pi_u M_{zz}^{-1}\widetilde{B}$ *and* $\mathbb{B}_s = \Pi_s M_{zz}^{-1}\widetilde{B}$*. Conversely, if* $\zeta_u = \Pi_u\mathbf{Z}$ *and* $\zeta_s = \Pi_s\mathbf{Z}$ *are solution of systems* (6.2.21) *and* (6.2.22)*, respectively, then* $\mathbf{Z}$ *is solution to* (6.2.20)*.*

To construct a stabilizing feedback law for (6.2.21), we consider the Riccati equation

$$\begin{cases} \mathbb{Q}_u \in \mathcal{L}(\mathbb{R}^{N_u}), \quad \mathbb{Q}_u = \mathbb{Q}_u^\top \geq 0, \\ \mathbb{Q}_u(\Xi_u + \omega_{sh}\mathbf{I}_{\mathbb{R}^{N_u}}) + (\Xi_u^\top + \omega_{sh}\mathbf{I}_{\mathbb{R}^{N_u}})\mathbb{Q}_u - \mathbb{Q}_u\widetilde{E}_u^\top\widetilde{B}\widetilde{B}^\top\widetilde{E}_u\mathbb{Q}_u = 0, \end{cases} \tag{6.2.23}$$

where ω_{sh} is a shift parameter. Then, the feedback law is given by

$$K = -\widetilde{B}^\top\widetilde{E}_u\mathbb{Q}_u\widetilde{E}_u^\top M_{zz}. \tag{6.2.24}$$

6.3 Algorithm for FSI direct problem

In this section, we recall the time-marching process introduced in Section 4.3 of Chapter 4.

We consider the ALE mapping $\mathcal{A}(t,\cdot) : \Omega \longrightarrow \Omega_{\eta(t)}$ defined by

$$\mathcal{A}(t,\cdot) = I + \int_0^t \mathbf{w}(s,\cdot)\, ds, \tag{6.3.1}$$

where $\mathbf{w}(t,\cdot)$ is solution of the elliptic equation

$$\Delta\mathbf{w} = 0 \ \text{ in } \Omega, \quad \mathbf{w} = \mathbf{u}|_{\Gamma_s} \ \text{ on } \Gamma_s, \quad \mathbf{w} = 0 \ \text{ on } \Gamma \setminus \Gamma_s, \tag{6.3.2}$$

where $\mathbf{u}|_{\Gamma_s}$ stands for the trace of the fluid velocity on Γ_s. Let Δt denote the time step, and define $t^k = k\Delta t$ for $k \in \mathbb{N}$, representing the time at level k. For all $k \in \mathbb{N}$, $\Omega^k := \mathcal{A}(t^k, \Omega_{ref})$ with boundary $\Gamma^k = \Gamma_i \cup \Gamma_0 \cup \Gamma_s^k \cup \Gamma_n$, where $\Gamma_0 = \Gamma_r \cup \Gamma_w$ and $\Gamma_s^k = \Gamma_{s,top}^k \cup \Gamma_{s,bot}^k \cup \Gamma_{s,lat}^k$. We denote by $\mathbf{u}^k$, p^k and $\boldsymbol{\lambda}^k$ the approximations of $\mathbf{u}(t^k,\cdot)$, $p(t^k,\cdot)$ and $\boldsymbol{\lambda}(t^k,\cdot)$, respectively. Here, $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}_i^k, \boldsymbol{\lambda}_0^k, \boldsymbol{\lambda}_{s,top}^k, \boldsymbol{\lambda}_{s,bot}^k, \boldsymbol{\lambda}_{s,lat}^k)^\top$ denotes the Langrange multiplier associated to the Dirichlet boundary conditions. We also denote by η_1^k and η_2^k the approximation of $\eta_1(t^k,\cdot)$ and $\eta_2(t^k,\cdot)$ defined on $(0, \ell_s)$, respectively.

Then, assuming known $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k and η_2^k, let us consider the following problem:

Find $\widehat{\mathbf{u}}^{k+1} \in \mathbf{H}^1(\Omega^k)$, $\widehat{p}^{k+1} \in L^2(\Omega^k)$, $\widehat{\boldsymbol{\lambda}}^{k+1} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma^k \setminus \Gamma_n)$, $\eta_1^{k+1}, \eta_2^{k+1} \in H^2_{\{0\}}(0, \ell_s)$ such that

$$\left\{\begin{aligned}
&\int_{\Omega^k} \frac{\widehat{\mathbf{u}}^{k+1} - \mathbf{u}^k}{\Delta t} \cdot \boldsymbol{\phi} = a_f(\widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) + b(\boldsymbol{\phi}, \widehat{p}^{k+1}) + c(\widehat{\mathbf{u}}^{k+1} - \mathbf{w}^k, \widehat{\mathbf{u}}^{k+1}, \boldsymbol{\phi}) \\
&\qquad + \int_{\Gamma_i} \widehat{\boldsymbol{\lambda}}_i^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma_0^k} \widehat{\boldsymbol{\lambda}}_0^{k+1} \cdot \boldsymbol{\phi} + \int_{\Gamma_{s,top}^k} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \boldsymbol{\phi} \\
&\qquad + \int_{\Gamma_{s,bot}^k} \widehat{\boldsymbol{\lambda}}_{s,bot}^{k+1} \cdot \boldsymbol{\phi}, + \int_{\Gamma_{s,lat}^k} \widehat{\boldsymbol{\lambda}}_{s,lat}^{k+1} \cdot \boldsymbol{\phi}, \quad \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega^k), \\
&b(\widehat{\mathbf{u}}^{k+1}, \psi) = 0, \quad \forall \psi \in L^2(\Omega^k), \\
&\int_{\Gamma_i} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_i} \mathbf{g}^i \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i), \qquad \int_{\Gamma_0} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_0), \\
&\int_{\Gamma_{s,top}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,top}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,top}^k), \\
&\int_{\Gamma_{s,bot}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,bot}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,bot}^k), \\
&\int_{\Gamma_{s,lat}^k} \widehat{\mathbf{u}}^{k+1} \cdot \boldsymbol{\tau} = \int_{\Gamma_{s,lat}^k} \eta_2^{k+1} \vec{\mathbf{e}}_2 \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{s,lat}^k), \\
&\int_0^{\ell_s} \frac{\eta_1^{k+1} - \eta_1^k}{\Delta t} \zeta = \int_0^{\ell_s} \eta_2^{k+1} \zeta, \quad \forall \zeta \in H^2_{\{0\}}(0, \ell_s), \\
&\int_0^{\ell_s} \frac{\eta_2^{k+1} - \eta_2^k}{\Delta t} \zeta = a_s^1(\eta_1^{k+1}, \zeta) + a_s^2(\eta_2^{k+1}, \zeta) \\
&\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{s,top}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} \\
&\qquad - \int_0^{\ell_s} \widehat{\boldsymbol{\lambda}}_{bot}^{k+1} \cdot \vec{\mathbf{e}}_2 \zeta \sqrt{1 + (\eta_{1,x}^k)^2} \\
&\qquad + \sum_{j=1}^{N_c} \int_0^{\ell_s} w_j(z_1) \zeta(z_1) f_j^k(t) + \int_0^{\ell_s} f_s \zeta, \quad \forall \zeta \in H^2_{\{0\}}(0, \ell_s).
\end{aligned}\right. \tag{6.3.3}$$

where

$$a_f(\mathbf{v}, \boldsymbol{\phi}) = -2\nu \int_{\Omega^k} \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\phi}), \quad b(\boldsymbol{\phi}, q) = \int_{\Omega^k} \operatorname{div} \boldsymbol{\phi}\, q, \quad c(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}) = -\int_{\Omega^k} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \boldsymbol{\phi},$$
$$a_s^1(\eta_1, \zeta) = -\alpha \int_0^{\ell_s} \Delta \eta_1 \cdot \Delta \zeta, \quad a_s^2(\eta_2, \zeta) = -\gamma \int_0^{\ell_s} \Delta \eta_2 \cdot \Delta \zeta.$$

Given the solution at the instant t^k, $\mathbf{u}^k$, p^k, $\boldsymbol{\lambda}^k$, $\mathbf{w}^k$, η_1^k and η_2^k at the known configuration Ω^k, the procedure to solve the time advancing scheme from k to $k+1$ is described in Algorithm 4 below.

6.4 Numerical experiments

In this section, we present numerical experiments to analyze the performance of the linear feedback law when it is applied to the nonlinear system. More precisely, by fixing the Reynolds number $Re = 200$ and the damping coefficient $\gamma = 10^{-6}$, we consider different values of the rigidity coefficient α and various levels of the perturbation amplitude β. The geometrical parameters of the initial domain configuration Ω are described in Table 4.1 (see Figure 6.1).

Algorithm 4: Semi-implicit algorithm

For $k \geq 1$:

1 : Solve the linear systems that yields after applying the Newton algorithm in system (6.3.3) and get $\widehat{\mathbf{u}}^{k+1}$, $\widehat{p}^{k+1}$, $\widehat{\boldsymbol{\lambda}}^{k+1}$, η_1^{k+1}, η_2^{k+1}.

2 : Compute the mesh velocity $\widehat{\mathbf{w}}^{k+1} : \Omega^k \to \mathbb{R}^2$ satisfying the elliptic equation

$$\begin{cases} \Delta \widehat{\mathbf{w}}^{k+1} = 0 \ \text{ in } \Omega^k, \\ \widehat{\mathbf{w}}^{k+1} = \widehat{\mathbf{u}}^{k+1} \ \text{ on } \Gamma_s^k, \\ \widehat{\mathbf{w}}^{k+1} = 0 \ \text{ on } \Gamma \setminus \Gamma_s^k. \end{cases} \tag{6.3.4}$$

3 : Define $\mathcal{A}^k(\widehat{\mathbf{x}}) := \widehat{\mathbf{x}} + \Delta t \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}})$ and $\Omega^{k+1} := \mathcal{A}^k(\Omega^k)$.

4 : Define $\mathbf{u}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$, $p : \Omega^{k+1} \to \mathbb{R}$, $\boldsymbol{\lambda}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ and $\mathbf{w}^{k+1} : \Omega^{k+1} \to \mathbb{R}^2$ by

$$\begin{aligned} &\mathbf{u}^{k+1}(\mathbf{x}) = \widehat{\mathbf{u}}^{k+1}(\widehat{\mathbf{x}}), \ \ p^{k+1}(\mathbf{x}) = \widehat{p}^{k+1}(\widehat{\mathbf{x}}), \ \ \boldsymbol{\lambda}^{k+1}(\mathbf{x}) = \widehat{\boldsymbol{\lambda}}^{k+1}(\widehat{\mathbf{x}}) \\ &\text{and } \ \mathbf{w}^{k+1}(\mathbf{x}) = \widehat{\mathbf{w}}^{k+1}(\widehat{\mathbf{x}}), \ \ \forall \mathbf{x} = \mathcal{A}^k(\widehat{\mathbf{x}}), \, \widehat{\mathbf{x}} \in \Omega^k. \end{aligned}$$

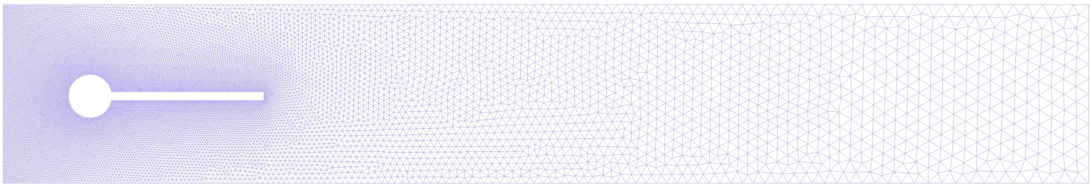


Figure 6.1 – Geometrical configuration and triangular mesh used in the numerical simulation.

As in Section 4.4 of Chapter 4, we use a triangular mesh with 30168 cells locally refined around the structure, see Figure 6.1. For the space discretization of system (6.3.3), we choose the generalized Taylor-Hood finite elements $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$ for the velocity, the pressure and the Lagrange multiplier, respectively. The displacement and the velocity of the structure are discretized by using Hermite finite elements. The nonlinearity is treated with a Newton algorithm. The total of degree of freedom is equal to 394803. All numerical simulation of the unsteady system were carried out using a time step $\Delta t = 5 \cdot 10^{-4}$.

In order to facilitate the analysis, for each of the three values of the rigidity coefficient α ($\alpha = 1, 10^{-1}, 10^{-2}$) and when the Reynolds number $Re = 200$ and the damping coefficient $\gamma = 10^{-6}$, we first compare the spectrum of the fluid-structure operator with and without feedback. Secondly, for a fixed pertubation amplitude β, we analyze the influence of the choice of the unstable subspace $\mathbb{Z}_u$ by considering two alternatives: the unstable subspace $\mathbb{Z}_u^1$, associated with the unstable eigenvalues, and the extended subspace $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_s)$, where $G(\mu_s)$ denotes the subspace associated to the first stable eigenvalue μ_s (ordered according to the real parts). Although other choices are possible, these cases were selected for the sake of simplicity. Finally, we investigate the performance of the feedback laws under the influence of the perturbation amplitude β.

We remark that, due to the deterioration of the mesh when $\alpha = 10^{-3}$ (see Figure 4.34), we do not include analysis of this case, as it prevents a meaningful comparison between the controlled and uncontrolled scenarios.

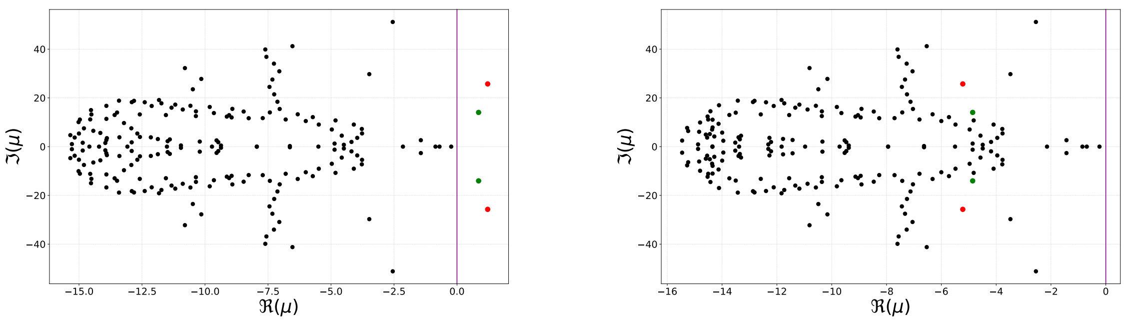
6.4.1 Case 1: Rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, $Re = 200$ and $\gamma = 10^{-6}$

- *Comparison between the spectra of the fluid-structure operator with and without feedback.*

Table 6.1 presents the unstable eigenvalues of the fluid-structure operator A, which are denoted by μ, along with the corresponding eigenvalues of the controlled operator $A - BK$, denoted by $\widetilde{\mu}$, where the feedback operator K is defined in (6.2.24). A graphical representation is provided in Figure 6.2b, where a comparison between portions of the spectra of both operators is presented.

$\boldsymbol{\mu}_{1,2}$	$\widetilde{\boldsymbol{\mu}}_{1,2}$	$\boldsymbol{\mu}_{3,4}$	$\widetilde{\boldsymbol{\mu}}_{3,4}$
$1.21 \pm 25.72i$	$-5.21 \pm 25.72i$	$0.85 \pm 14.03i$	$-4.85 \pm 14.03i$

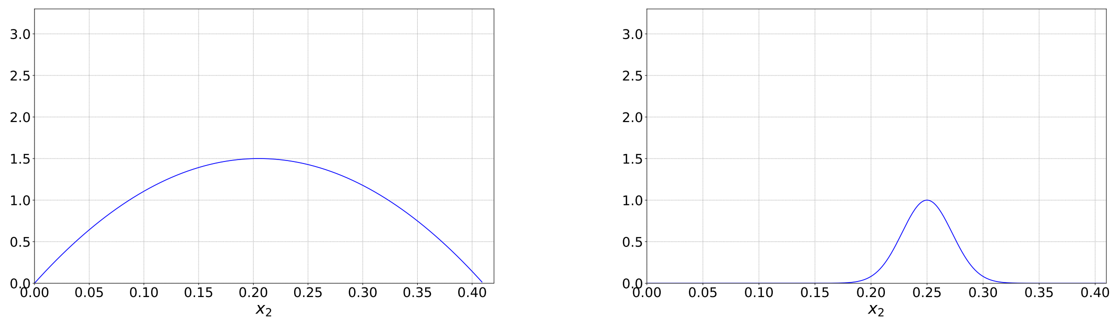
Table 6.1 – Eigenvalues of the fluid-structure system without feedback (denoted by μ) and with feedback (denoted by $\widetilde{\mu}$) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, and damping coefficient $\gamma = 10^{-6}$. The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ with a shift coefficient $\omega_{sh} = 2$.



(a) Spectrum of the operator A.

(b) Spectrum of the operator $A - BK$.

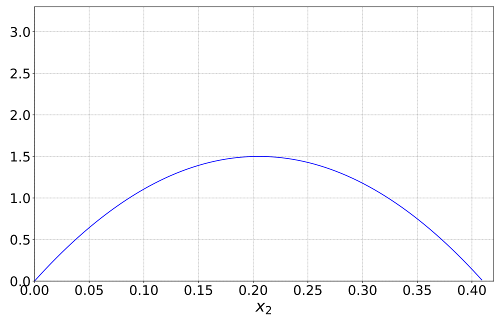
Figure 6.2 – Comparison of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, and damping coefficient $\gamma = 10^{-6}$, with and without control. The unstable eigenvalues of the operator A (denoted by μ) and those of the operator $A - BK$ (denoted by $\widetilde{\mu}$) are colored in red (the conjugate pairs $\mu_{1,2}$ and $\widetilde{\mu}_{1,2}$) and green (the conjugate pairs $\mu_{3,4}$ and $\widetilde{\mu}_{3,4}$). The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ with a shift coefficient $\omega_{sh} = 2$.



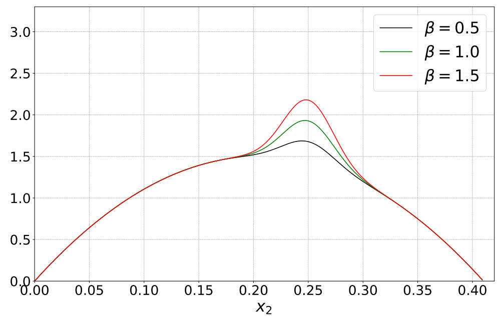
(a) Profile of the inflow condition $g_{s,1}^i$.

(b) Profile perturbation g_p^i at $t = 0$.

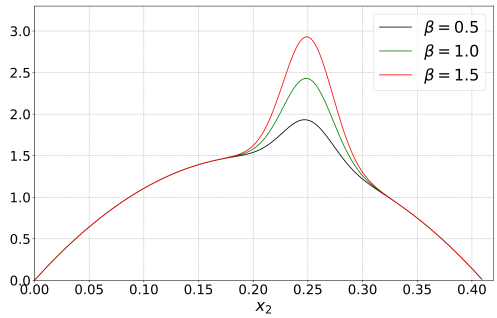
Figure 6.3 – Profile of the inflow condition $g_{s,1}^i$ and perturbation g_p^i.



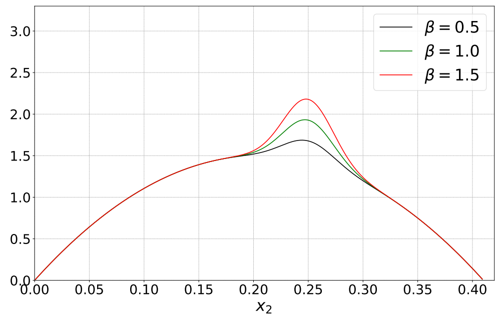
(a) $t = 0[s]$ (without perturbation)



(b) $t = 0.5[s]$



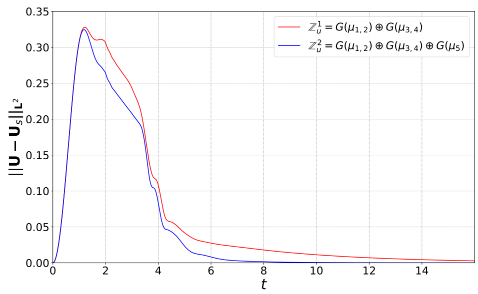
(c) $t = 1.0[s]$



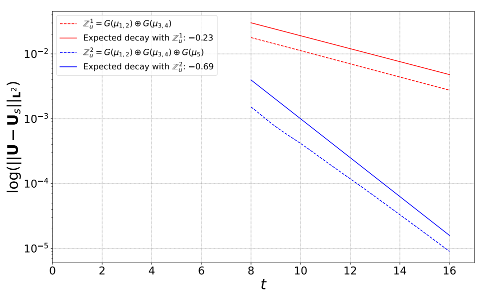
(d) $t = 1.5[s]$

Figure 6.4 – Profile of the first component of the inflow data $\mathbf{g}^i$ at different time instants for different values of the perturbation parameter β.

- *Influence of the unstable subspace* $\mathbb{Z}_u$. In order to compute the feedback law, we consider two unstable subspaces $\mathbb{Z}_u$: $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$, where $\mu_5 = -0.23$. To carry out the numerical simulations, we set the perturbation amplitude β to 1.5 (see Figure 6.4). Figure 6.5b corroborates the expected decay rate in each case. Furthermore, as is expected, the best decay rate is obtained in the case when $\mathbb{Z}_u = \mathbb{Z}_u^2$ (see Figure 6.5a).

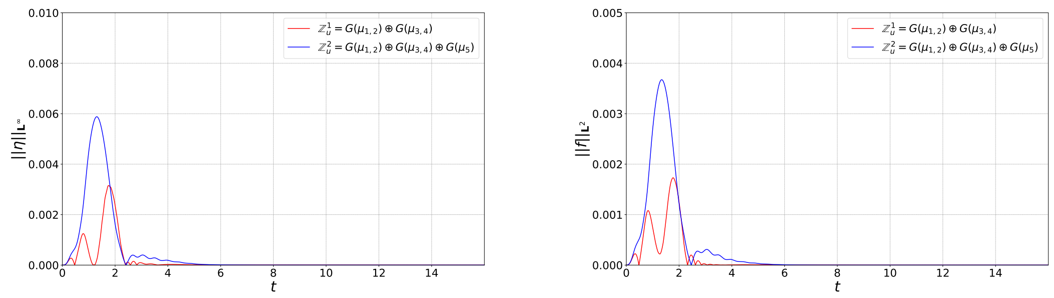


(a) Time evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$.



(b) Time evolution of $\log(\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2})$.

Figure 6.5 – Comparison between the decay rates when $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, in the case of rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.



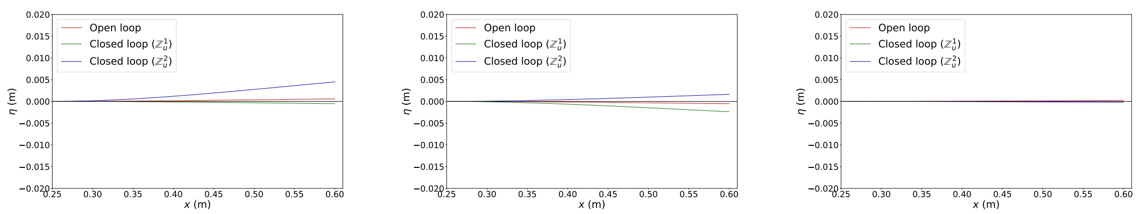
(a) Time evolution of $\|\eta\|_{L^\infty}$.

(b) Time evolution of $\|f\|_{L^2}$.

Figure 6.6 – Time evolution of $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$ for the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, when the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and the perturbation amplitude $\beta = 1.5$.

We first observe that in the case of the feedback law based on $\mathbb{Z}_u^1$, the control mainly acts during the first two seconds (see red curve in Figure 6.6b). This coincides with the period during which the structure experiences the most significant displacements (see red curve in Figure 6.6a), which is expected given that the control acts on the structure. Similarly, when the feedback law control is based on $\mathbb{Z}_u = \mathbb{Z}_u^2$, Figure 6.6b (see blue curve) shows that the control action is mainly concentrated during the first two seconds. As in the case $\mathbb{Z}_u^1$, during this interval of time we observe that the structure experiences the most significant displacements (see blue curve in Figure 6.6a). Furthermore, over the same period, the magnitude of the control is greater compared to that associated to $\mathbb{Z}_u^1$, as shown in Figure 6.6b.

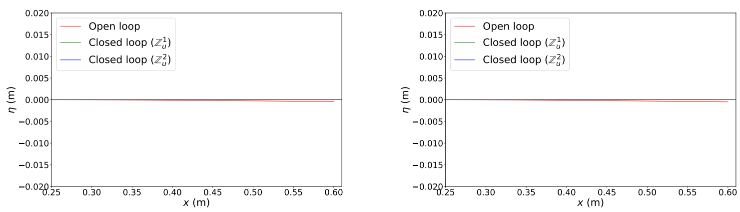
Figure 6.7 shows a sequence of snapshots illustrating the evolution of the structure's deflection at different time instants, considering both the open-loop and closed-loop systems (feedback control law based on $\mathbb{Z}_u^1$ and $\mathbb{Z}_u^2$), when the perturbation amplitude is fixed at $\beta = 1.5$. From Figures 6.7c, 6.7d and 6.7e, we observe that the displacement of the structure is stabilized around $\eta = 0$, which is also corroborated by the information displayed in Figures 6.11b and 6.11e.



(a) $t = 1[s]$

(b) $t = 2[s]$

(c) $t = 4[s]$



(d) $t = 11[s]$

(e) $t = 15[s]$

Figure 6.7 – Snapshots of the structure's deflection at different time instants, without and with control (based on the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$) corresponding to the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and the perturbation amplitude $\beta = 1.5$.

In Figure 6.9, we compare the magnitude of the fluid velocity in the open loop system, and the closed loop system where the feedback law is based on the unstable subspace $\mathbb{Z}_u = \mathbb{Z}_u^1$. As observed at time instants $t = 1[s]$ and $t = 2[s]$, the global behavior appears to be similar in both cases. However, from $t = 4[s]$ onward, we notice that the fluid velocity stabilizes around the stationary fluid velocity (see also Figure 6.8). Similar conclusions are drawn when the feedback law is based on the unstable subspace $\mathbb{Z}_u^2$ (see Figure 6.10).

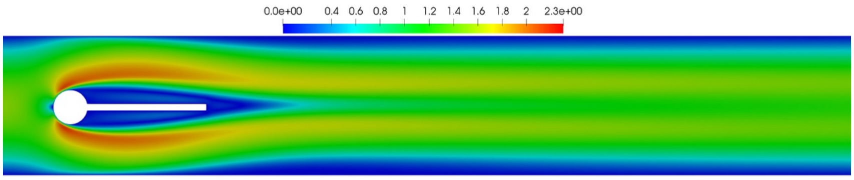
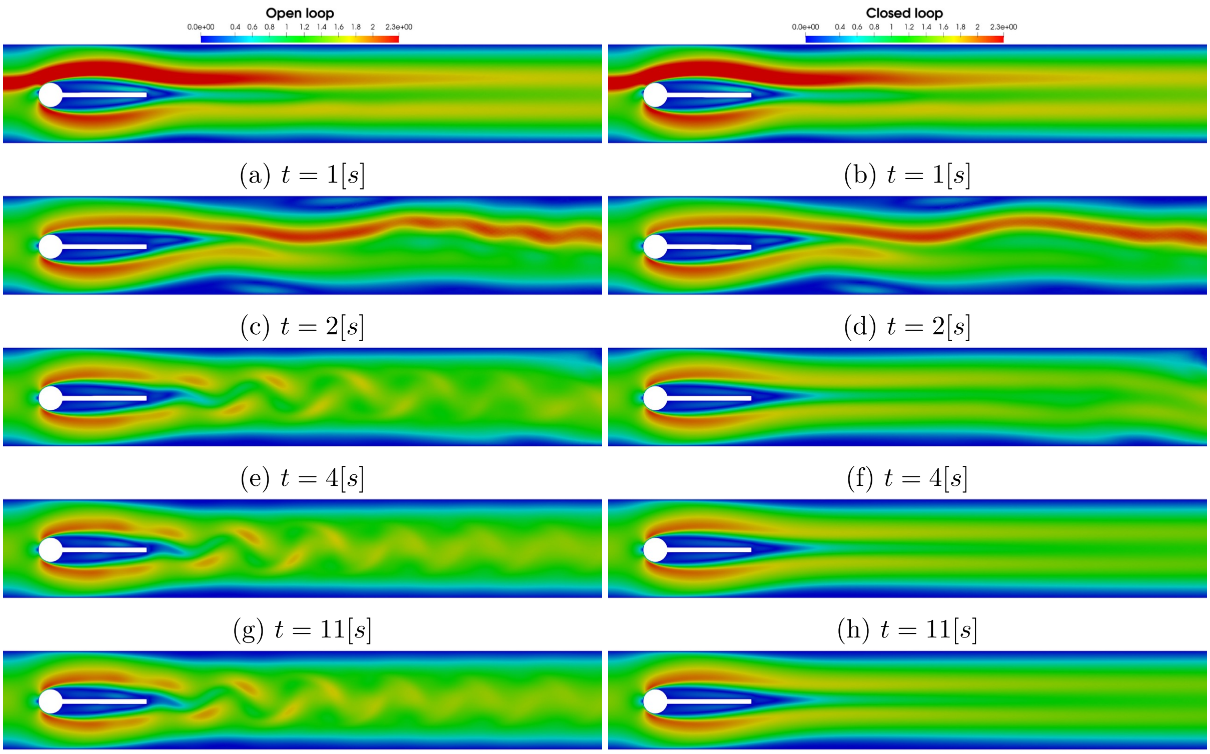


Figure 6.8 – Fluid velocity magnitude $\mathbf{U}_s$ corresponding to Reynolds number $Re = 200$.



(a) $t = 1[s]$ (b) $t = 1[s]$

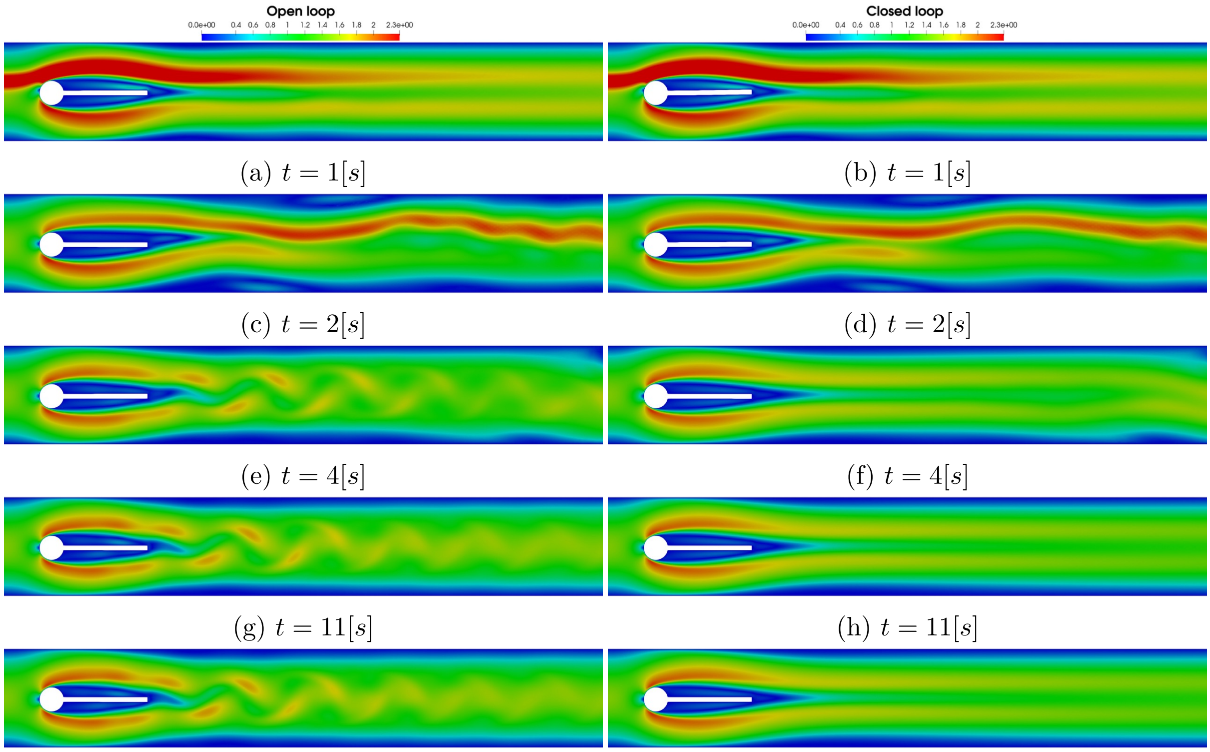
(c) $t = 2[s]$ (d) $t = 2[s]$

(e) $t = 4[s]$ (f) $t = 4[s]$

(g) $t = 11[s]$ (h) $t = 11[s]$

(i) $t = 15[s]$ (j) $t = 15[s]$

Figure 6.9 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{1}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$. In the left column (a)-(c)-(e)-(g)-(i), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j), we display the velocity magnitudes when the control is applied.



(a) $t = 1[s]$ (b) $t = 1[s]$

(c) $t = 2[s]$ (d) $t = 2[s]$

(e) $t = 4[s]$ (f) $t = 4[s]$

(g) $t = 11[s]$ (h) $t = 11[s]$

(i) $t = 15[s]$ (j) $t = 15[s]$

Figure 6.10 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 1}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$. In the left column (a)-(c)-(e)-(g)-(i)-(k), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j)-(l), we display the velocity magnitudes when the control is applied.

- *Influence of the perturbation amplitude β.* We analyze the performance of the feedback control law based on each of the two unstable subspaces $\mathbb{Z}_u$. More precisely, in order to get a first glance of the performance of the control, we plot the evolution of the norms $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$, for three given values of the perturbation amplitude β: 0.5, 1, 1.5.

	$\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 4$)			$\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^2) = 5$)		
β	$\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$	$\|\eta\|_{L^\infty}$	$\|f\|_{L^2}$	$\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$	$\|\eta\|_{L^\infty}$	$\|f\|_{L^2}$
0.5	10^{-3}	$5 \cdot 10^{-7}$	$6 \cdot 10^{-9}$	$5 \cdot 10^{-6}$	$2 \cdot 10^{-10}$	$7 \cdot 10^{-11}$
1	$2 \cdot 10^{-3}$	10^{-6}	$2 \cdot 10^{-8}$	$9 \cdot 10^{-6}$	$6 \cdot 10^{-10}$	$2 \cdot 10^{-10}$
1.5	$3 \cdot 10^{-3}$	10^{-6}	$4 \cdot 10^{-8}$	10^{-5}	$9 \cdot 10^{-10}$	$2 \cdot 10^{-10}$

Table 6.2 – Order of quantities $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$ at $t = 15.5[s]$, for two different choices of the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, and for three different values of the perturbation amplitude β. The rigidity coefficient is fixed at $\boldsymbol{\alpha = 1}$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$.

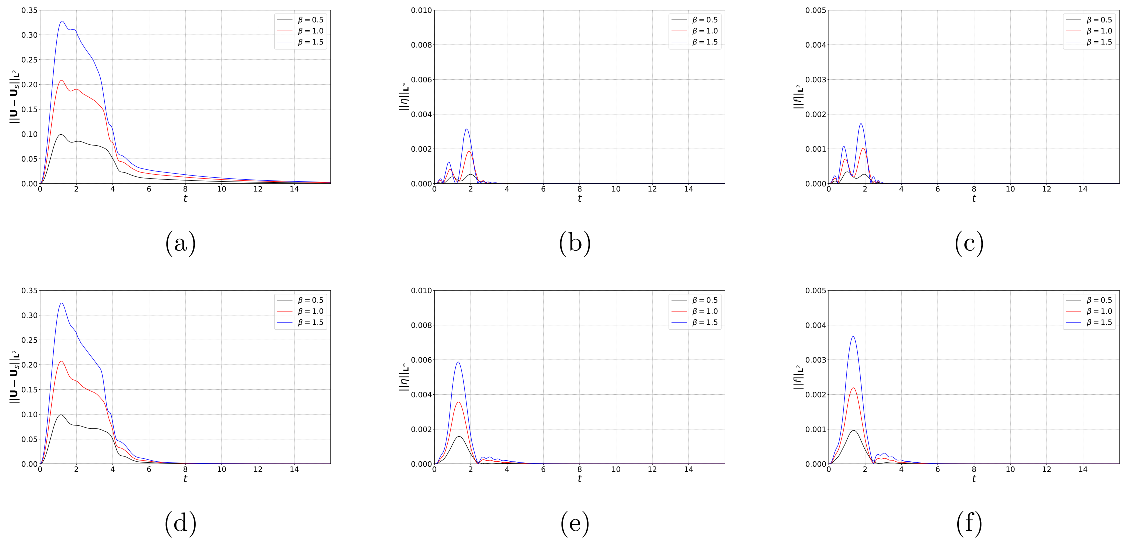


Figure 6.11 – Evolution of (a): $\|\mathbf{U}-\mathbf{U}_s\|_{\mathbf{L}^2}$, (b): $\|\eta\|_{L^\infty}$ and (c): $\|f\|_{L^2}$, for three different values of the amplitude perturbation β, when the rigidity coefficient $\alpha = 1$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$. Row (a)-(b)-(c): Control based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$. Row (d)-(e)-(f): Control based on $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$.

An initial expected conclusion that it is possible to deduce from Figures 6.11c and 6.11f, is that the magnitude of the control grows with increasing perturbation amplitude β. The effect of the size of the perturbation amplitude is also observed in the evolution of the quantities $\|\mathbf{U}-\mathbf{U}_s\|_{\mathbf{L}^2}$ and $\|\eta\|_{L^\infty}$, as show Figures 6.11a, 6.11d, and 6.11b, 6.11e, respectively. Table 6.2 shows the orders of magnitudes of the quantities $\|\mathbf{U}-\mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$, and $\|f\|_{L^2}$ at time $t = 15.5[s]$. This table allows us to observe the difference of the orders of magnitudes between the feedback control based on $\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 4$) and $\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^2) = 5$). This information supports the claim that using $\mathbb{Z}_u^2$ instead of $\mathbb{Z}_u^1$ improves the stabilization performance.

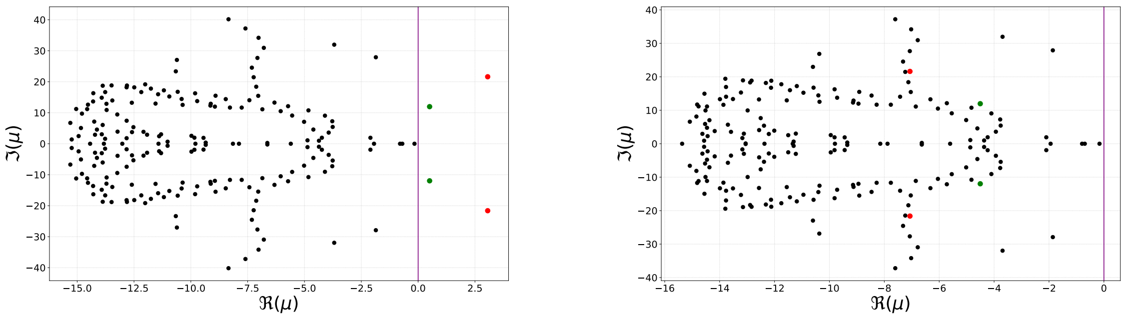
6.4.2 Case 2: Rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-1}}$, $Re = 200$ and $\gamma = 10^{-6}$

- *Comparison between the spectra of the fluid-structure operator with and without feedback.*

Figure 6.12b shows how the spectrum is modified after applying the feedback law. Figure 6.12a displays the spectrum of the fluid-structure operator without the action of the feedback operator. As complementary information, Table 6.3 shows the values of the unstable eigenvalues, along with their corresponding modifications after the feedback law is applied.

$\boldsymbol{\mu}_{1,2}$	$\widetilde{\boldsymbol{\mu}}_{1,2}$	$\boldsymbol{\mu}_{3,4}$	$\widetilde{\boldsymbol{\mu}}_{3,4}$
$3.05 \pm 21.61i$	$-7.06 \pm 21.61i$	$0.50 \pm 11.96i$	$-4.50 \pm 11.96i$

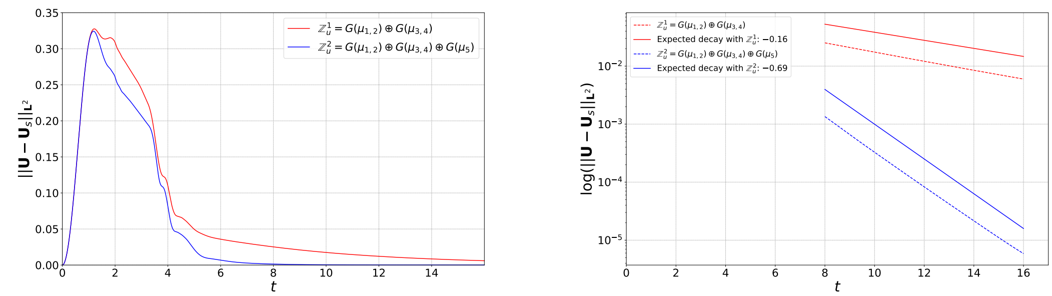
Table 6.3 – Eigenvalues of the fluid-structure system without feedback (μ) and with feedback ($\widetilde{\mu}$) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-1}}$, and damping coefficient $\gamma = 10^{-6}$. The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ with a shift coefficient $\omega_{sh} = 2$.



(a) Spectrum of the operator A. (b) Spectrum of the operator $A - BK$.

Figure 6.12 – Comparison of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-1}}$, and damping coefficient $\gamma = 10^{-6}$, with and without control. The unstable eigenvalues of the operator A (denoted by μ) and those of the operator $A - BK$ (denoted by $\widetilde{\mu}$) are colored in red (the conjugate pairs $\mu_{1,2}$ and $\widetilde{\mu}_{1,2}$) and green (the conjugate pairs $\mu_{3,4}$ and $\widetilde{\mu}_{3,4}$). The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ with a shift coefficient $\omega_{sh} = 2$.

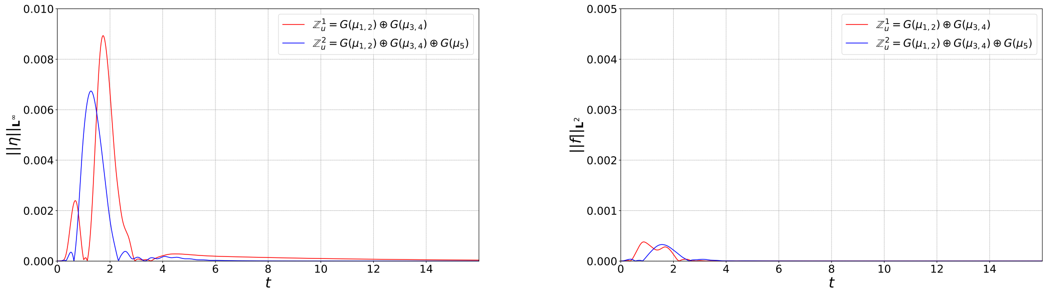
• *Influence of the unstable subspace $\mathbb{Z}_u$*. As expected, from Figure 6.13a we confirm that the control based on the unstable subspace $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$, where $\mu_5 = -0.16$, yields a better decay rate compared to the control based on the unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$, when the amplitude of the perturbation is fixed at $\beta = 1.5$. Moreover, the expected value of the decay rates in each case are corroborated by the information displayed in the semi-logarithmic plot shown in Figure 6.13b.



(a) Time evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$ (b) Time evolution of $\log(\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2})$

Figure 6.13 – Comparison between the decay rates when $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, in the case of rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-1}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.

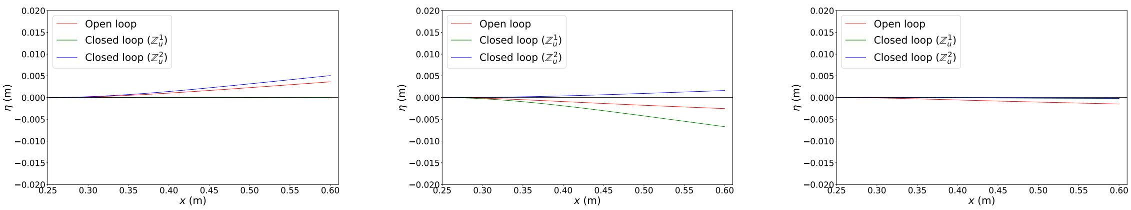
As in the previously analyzed case with $\alpha = 1$, Figures 6.14a and 6.14b, which respectively show the evolution of the quantities $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$, allow us to conclude that the period of significant activation of the control approximately coincides with the period of most pronounced displacement of the structure.



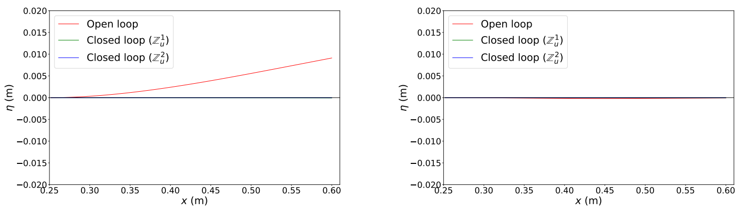
(a) Time evolution of $\|\eta\|_{L^\infty}$ (b) Time evolution of $\|f\|_{L^2}$

Figure 6.14 – Time evolution of $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$ for the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, when the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and the perturbation amplitude $\beta = 1.5$.

Figure 6.15 show snapshots of the evolution of the structure's displacement at different time instants, when $\beta = 1.5$. From these plots we observe the stabilization of the displacement of the structure around $\eta = 0$ in both cases. On the other hand, concerning the behavior of the fluid velocity, Figures 6.16 and 6.17, which show a comparison between the open loop system and the closed loop system based on the unstable subspaces $\mathbb{Z}_u^1$ and $\mathbb{Z}_u^2$, respectively, corroborate the insight mentioned above. Specifically, they show that the velocity fluid is stabilized around the stationary solution (see Figure 6.8). These results are similar to those obtained in the case when $\alpha = 1$. The main difference is that the structure exhibits larger deflection when $\alpha = 0.1$, as can be observed by comparing Figures 6.15 and 6.7.

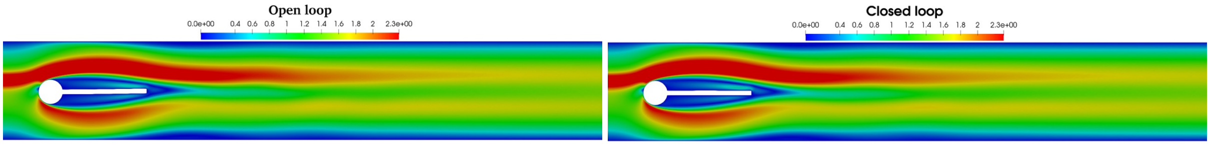


(a) $t = 1[s]$ (b) $t = 2[s]$ (c) $t = 4[s]$

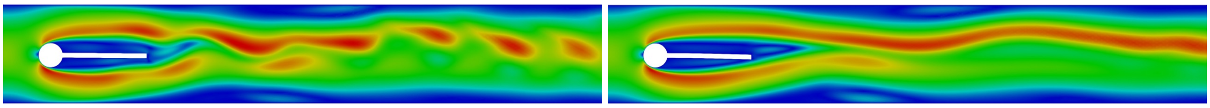


(d) $t = 11[s]$ (e) $t = 15[s]$

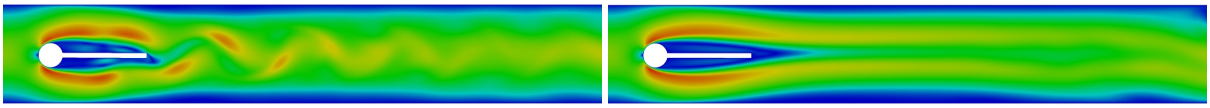
Figure 6.15 – Snapshots of the structure's deflection at different time instants, without and with control (based on the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$ corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.



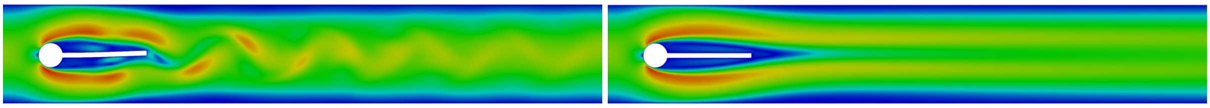
(a) $t = 1[s]$ (b) $t = 1[s]$



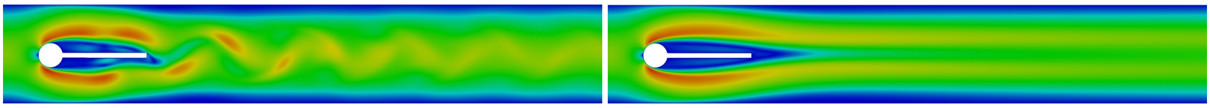
(c) $t = 2[s]$ (d) $t = 2[s]$



(e) $t = 4[s]$ (f) $t = 4[s]$

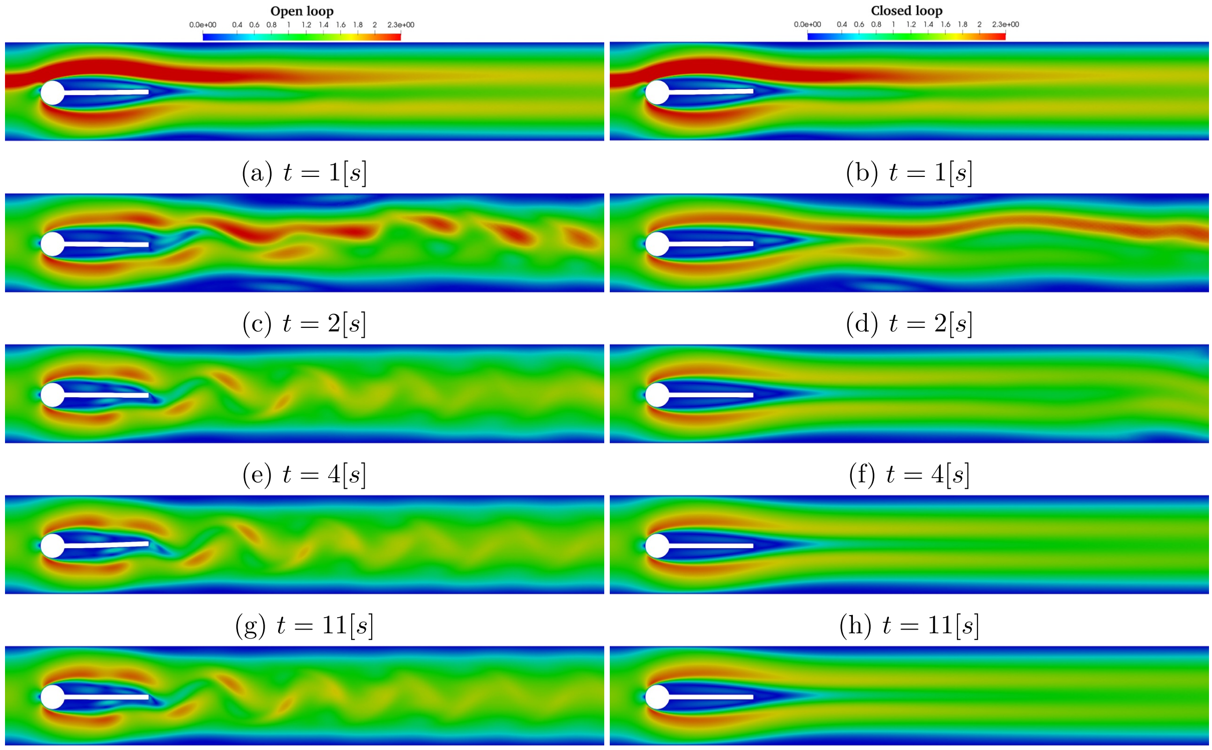


(g) $t = 11[s]$ (h) $t = 11[s]$



(i) $t = 15[s]$ (j) $t = 15[s]$

Figure 6.16 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$. In the left column (a)-(c)-(e)-(g)-(i), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j), we display the velocity magnitudes when the control is applied.



(a) $t = 1[s]$ (b) $t = 1[s]$

(c) $t = 2[s]$ (d) $t = 2[s]$

(e) $t = 4[s]$ (f) $t = 4[s]$

(g) $t = 11[s]$ (h) $t = 11[s]$

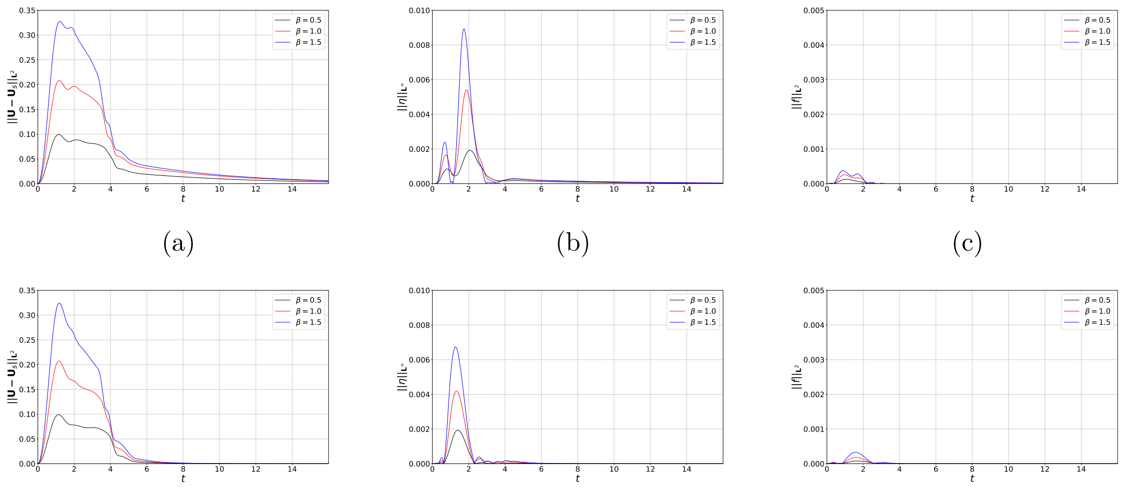
(i) $t = 15[s]$ (j) $t = 15[s]$

Figure 6.17 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-1}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$. In the left column (a)-(c)-(e)-(g)-(i), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j), we display the velocity magnitudes when the control is applied.

- *Influence of the perturbation amplitude* β. An in the case of $\alpha = 1$, we analyze the performance of the feedback control based on the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_{3,4}) \oplus G(\mu_5)$. The conclusions in this case are similar to those obtained for the case $\alpha = 1$. First, from Figure 6.18, we observe the influence of the perturbation amplitude β on the quantities $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$. Secondly, Table 6.4 is a complement to the information displayed in Figure 6.18 that supports the claim that the feedback operator based on $\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^1) = 5$) enhances the stabilization performance compared to the one based on $\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 4$).

	$\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 4$)			$\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^2) = 5$)		
β	$\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$	$\|\eta\|_{L^\infty}$	$\|f\|_{L^2}$	$\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$	$\|\eta\|_{L^\infty}$	$\|f\|_{L^2}$
0.5	$3 \cdot 10^{-3}$	$2 \cdot 10^{-5}$	10^{-8}	$3 \cdot 10^{-6}$	$5 \cdot 10^{-9}$	$9 \cdot 10^{-11}$
1	$5 \cdot 10^{-3}$	$3 \cdot 10^{-5}$	$3 \cdot 10^{-8}$	$5 \cdot 10^{-6}$	10^{-8}	10^{-10}
1.5	$6 \cdot 10^{-3}$	$4 \cdot 10^{-5}$	$3 \cdot 10^{-8}$	$8 \cdot 10^{-6}$	10^{-8}	$8 \cdot 10^{-11}$

Table 6.4 – Order of quantities $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$, $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$ at $t = 15.5[s]$, for two different choices of the unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$, and for three different values of the perturbation amplitude β. The rigidity coefficient is fixed at $\boldsymbol{\alpha = 10^{-1}}$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$.



(a) (b) (c)

(d) (e) (f)

Figure 6.18 – Evolution of (a): $\|\mathbf{U}-\mathbf{U}_s\|_{\mathbf{L}^2}$, (b): $\|\eta\|_{L^\infty}$ and (c): $\|f\|_{L^2}$, for three different values of the amplitude perturbation β, when the rigidity coefficient $\alpha = 10^{-1}$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$. Row (a)-(b)-(c): Control based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_{3,4})$. Row (d)-(e)-(f): Control based on $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_5)$.

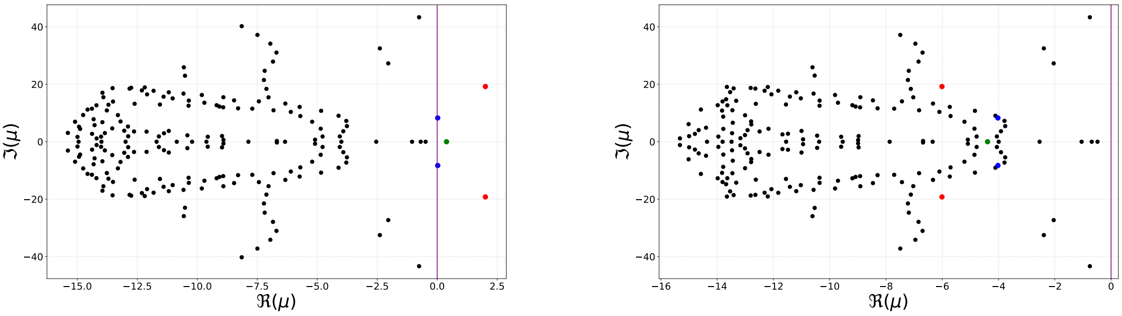
6.4.3 Case 3: Rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, $Re = 200$ and $\gamma = 10^{-6}$

- *Comparison between the spectra of the fluid-structure operator with and without feedback.*

In contrast to the two previously analyzed cases, which exhibited four unstable eigenvalues, the case with $\alpha = 10^{-2}$ exhibits five unstable eigenvalues. Table 6.5 reports those values (denoted by μ), together with the corresponding eigenvalues of the controlled system (denoted by $\widetilde{\mu}$) obtained by using the feedback law K based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ with a shift coefficient $\omega_{sh} = 2$. A graphical representation of a portion of the spectrum for the uncontrolled and controlled systems is presented in Figures 6.19a and 6.19b, respectively.

$\boldsymbol{\mu}_{1,2}$	$\widetilde{\boldsymbol{\mu}}_{1,2}$	$\boldsymbol{\mu}_3$	$\widetilde{\boldsymbol{\mu}}_3$	$\boldsymbol{\mu}_{4,5}$	$\widetilde{\boldsymbol{\mu}}_{4,5}$
$2.0 \pm 19.21i$	$-6.0 \pm 19.21i$	0.38	-4.38	$0.02 \pm 8.27i$	$-4.02 \pm 8.27i$

Table 6.5 – Eigenvalues of the fluid-structure system without feedback (μ) and with feedback ($\widetilde{\mu}$) corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$. The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ with a shift coefficient $\omega_{sh} = 2$.

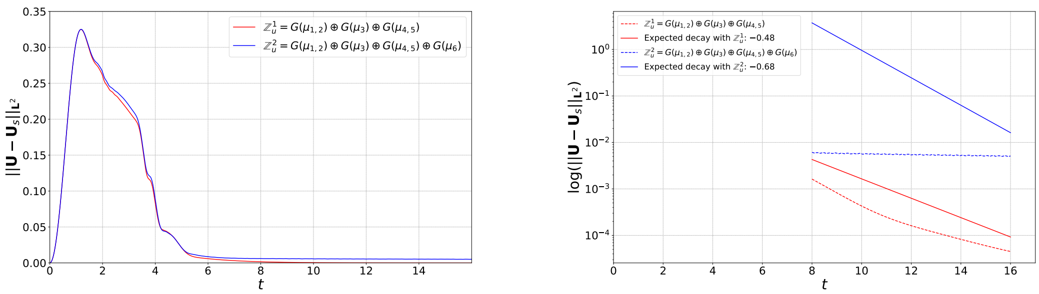


(a) Spectrum of the operator A.

(b) Spectrum of the operator $A - BK$.

Figure 6.19 – Comparison of the fluid-structure spectrum corresponding to Reynolds number $Re = 200$, rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-2}}$, and damping coefficient $\gamma = 10^{-6}$, with and without control. The unstable eigenvalues of the operator A (denoted by μ) and those of the operator $A - BK$ (denoted by $\widetilde{\mu}$) are colored in red (the conjugate pairs $\mu_{1,2}$ and $\widetilde{\mu}_{1,2}$), green (the real values μ_3 and $\widetilde{\mu}_3$) and blue (the conjugate pairs $\mu_{4,5}$ and $\widetilde{\mu}_{4,5}$). The feedback law K is based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ with a shift coefficient $\omega_{sh} = 2$.

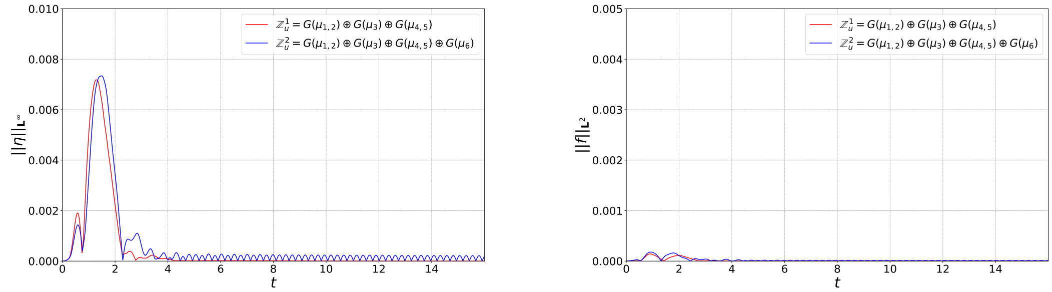
• <u>*Influence of the unstable subspace $\mathbb{Z}_u$*</u>. To compute the feedback law, we consider two unstable subspaces $\mathbb{Z}_u$: $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ and $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$, where $\mu_6 = -0.48$. The amplitude of the perturbation is fixed at $\beta = 1.5$. As visible in Figures 6.20a and 6.21a, the feedback control law based on the unstable subspace $\mathbb{Z}_u^1$ stabilizes the fluid velocity and the structure's displacement (see also Figures 6.23 (green curve) and 6.24). However, as shown in Figure 6.20b, the observed decay rate does not match the expected one, at least within the time period in which the numerical simulation was carried out. In fact, the results indicate that the decay rate is slower compared with the expected one.



(a) Time evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$

(b) Time evolution of $\log(\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2})$

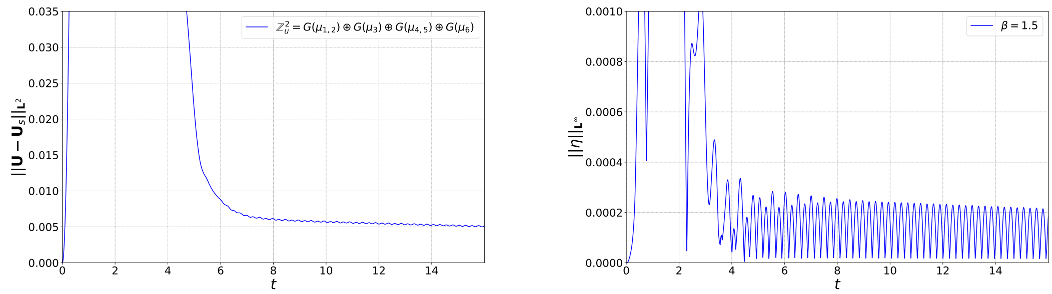
Figure 6.20 – Comparison between the decay rates when $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_6)$, in the case of rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.



(a) Time evolution of $\|\eta\|_{L^\infty}$ (b) Time evolution of $\|f\|_{L^2}$

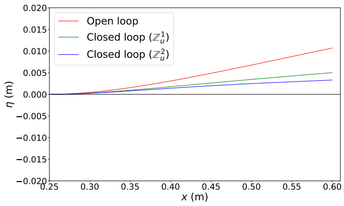
Figure 6.21 – Time evolution of $\|\eta\|_{L^\infty}$ and $\|f\|_{L^2}$ for the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_6)$, when the rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.

In the case of the feedback operator based on the unstable subspace $\mathbb{Z}_u^2$, with $\dim(\mathbb{Z}_u^2) = 6$, Figure 6.20a (curve in blue) is not sufficiently informative to determine whether the fluid velocity is stabilized. Figure 6.25, which displays snapshots of the fluid velocity magnitude, also does not provides this information. Figure 6.22a shows a zoomed-in view of the evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$. From this plot, we observe that, starting around $t = 7[s]$, the quantity $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$ decreases very slowly and exhibits an oscillatory behavior, which allows us to claim that the fluid velocity seems to be stabilized, but very weakly. Furthermore, as shown in Figure 6.20b, the observed decay rate is significantly lower than the expected one. On the other hand, it is not clear from Figure 6.21a whether the displacement of the structure is stabilized or not around $\eta = 0$. However, in the zoomed-in view presented in Figure 6.22b, it seems that the displacement of the structure is stabilized, although very weakly.

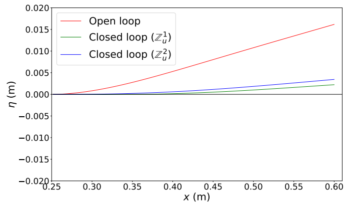


(a) (b)

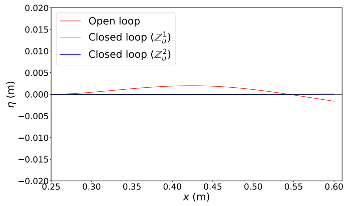
Figure 6.22 – Zoom-in view of evolution of (a) $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$ and (b) $\|\eta\|_{L^\infty}$, when $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$, in the case of rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.



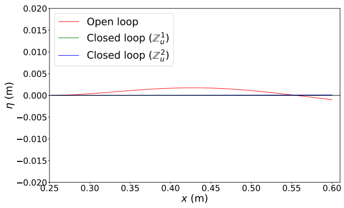
(a) $t = 1[s]$



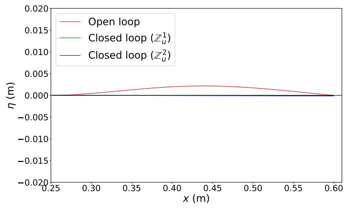
(b) $t = 2[s]$



(c) $t = 4[s]$

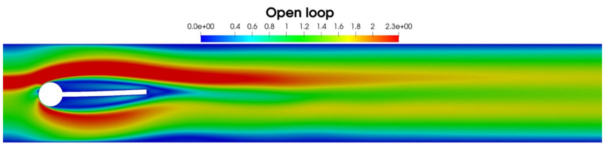


(d) $t = 11[s]$

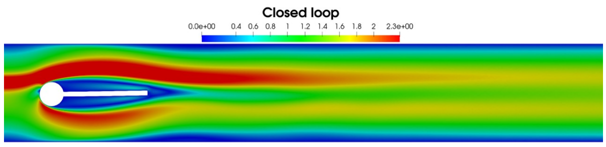


(e) $t = 15[s]$

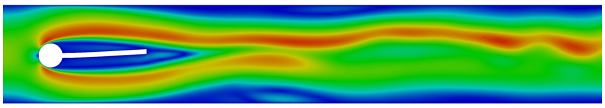
Figure 6.23 – Snapshots of the structure's deflection at different time instants, without and with control (based on the unstable subspaces $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_6)$ corresponding to the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1.5$.



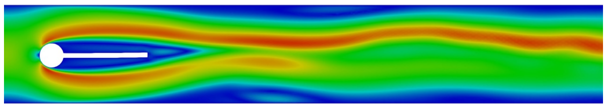
(a) $t = 1[s]$



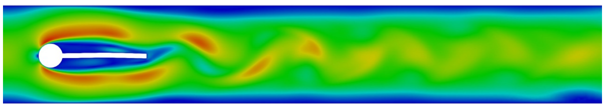
(b) $t = 1[s]$



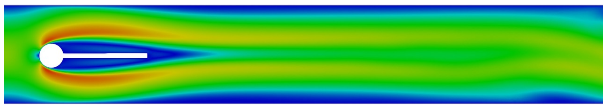
(c) $t = 2[s]$



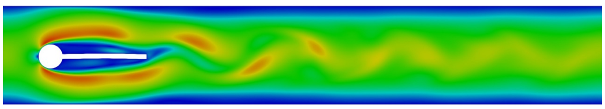
(d) $t = 2[s]$



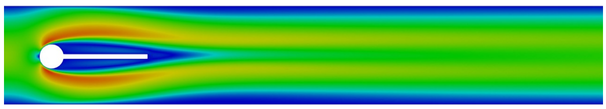
(e) $t = 4[s]$



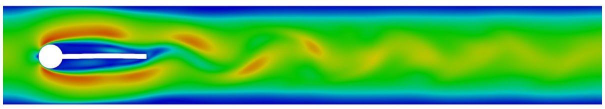
(f) $t = 4[s]$



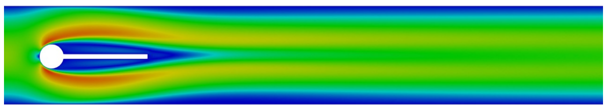
(g) $t = 11[s]$



(h) $t = 11[s]$

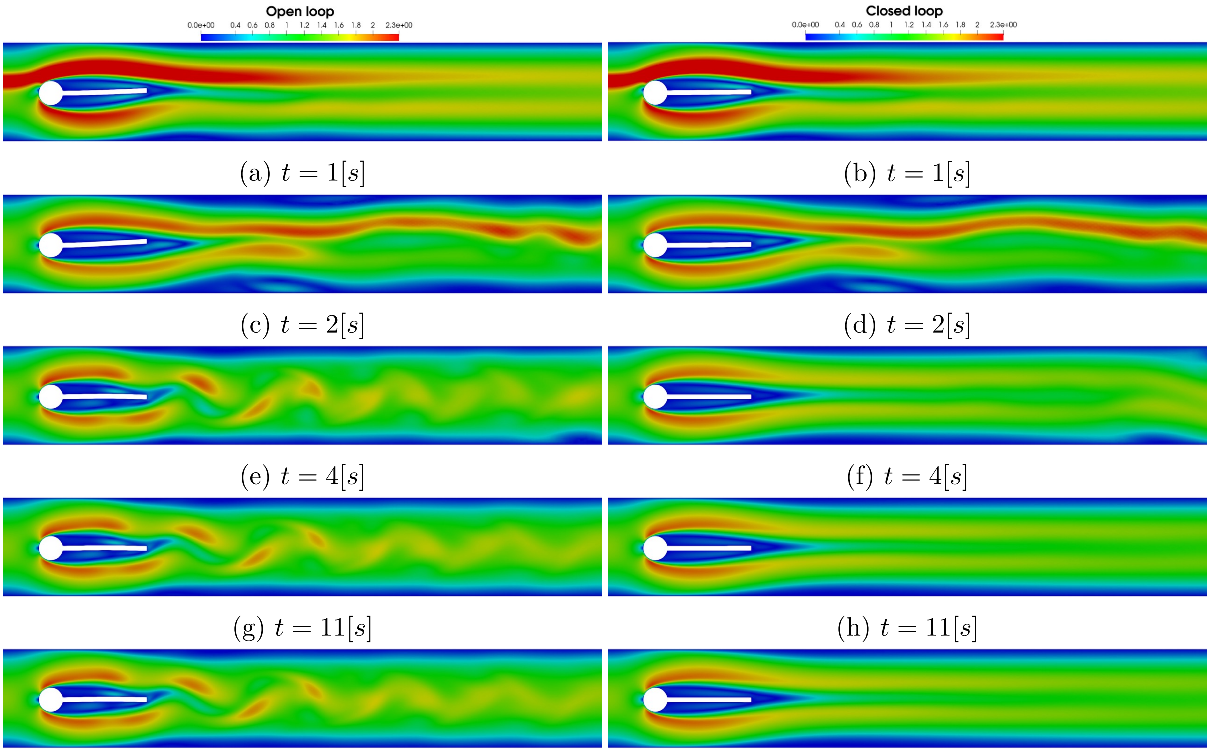


(i) $t = 15[s]$



(j) $t = 15[s]$

Figure 6.24 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha} = \mathbf{10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$. In the left column (a)-(c)-(e)-(g)-(i), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j), we display the velocity magnitudes when the control is applied.



(a) $t = 1[s]$ (b) $t = 1[s]$

(c) $t = 2[s]$ (d) $t = 2[s]$

(e) $t = 4[s]$ (f) $t = 4[s]$

(g) $t = 11[s]$ (h) $t = 11[s]$

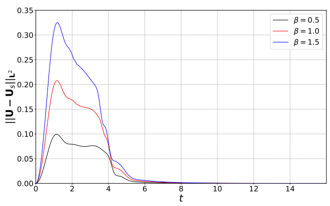
(i) $t = 15[s]$ (j) $t = 15[s]$

Figure 6.25 – Snapshots of the fluid velocity magnitude at different time instants, corresponding to the rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$, perturbation amplitude $\beta = 1.5$, and unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$. In the left column (a)-(c)-(e)-(g)-(i), we show the velocity magnitudes for the case without control, whereas in the right column (b)-(d)-(f)-(h)-(j), we display the velocity magnitudes when the control is applied.

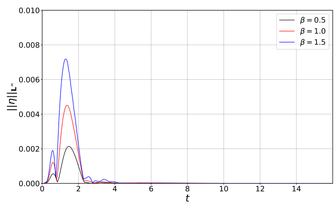
- *Influence of the perturbation amplitude β*. The plots shown in Figure 6.26 provide a first insight into the performance of the feedback operators based on $\mathbb{Z}_u^1$ and $\mathbb{Z}_u^2$ according to the three values of the perturbation amplitude β. In the case when the feedback operator is based on $\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 5$), the plots in Figures 6.26a and 6.26b show that the corresponding norms tend to zero for all three values of the perturbation amplitude. These results are also corroborated by the information displayed in Figures 6.23 and 6.24, which show snapshots of the structure's deflection and the magnitude of the fluid velocity, respectively, in the worst-case scenario with $\beta = 1.5$. However, as we pointed out above, in the worst-case scenario for the perturbation amplitude ($\beta = 1.5$), the observed decay rate does not match the expected one, at least within the time interval considered for the numerical simulation.

	$\mathbb{Z}_u^1$ ($\dim(\mathbb{Z}_u^1) = 5$)			$\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^2) = 6$)		
β	$\Vert\mathbf{U} - \mathbf{U}_s\Vert_{\mathbf{L}^2}$	$\Vert\eta\Vert_{L^\infty}$	$\Vert f\Vert_{L^2}$	$\Vert\mathbf{U} - \mathbf{U}_s\Vert_{\mathbf{L}^2}$	$\Vert\eta\Vert_{L^\infty}$	$\Vert f\Vert_{L^2}$
0.5	$2 \cdot 10^{-5}$	10^{-7}	10^{-10}	$9 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-6}$
1	$5 \cdot 10^{-5}$	$3 \cdot 10^{-7}$	$2 \cdot 10^{-9}$	$9 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-6}$
1.5	$5 \cdot 10^{-5}$	$2 \cdot 10^{-7}$	$3 \cdot 10^{-9}$	$5 \cdot 10^{-3}$	10^{-4}	10^{-5}

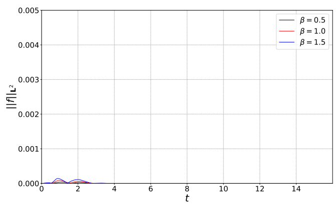
Table 6.6 – Order of quantities $\Vert\mathbf{U} - \mathbf{U}_s\Vert_{\mathbf{L}^2}$, $\Vert\eta\Vert_{L^\infty}$ and $\Vert f\Vert_{L^2}$ at $t = 15.5[s]$, for two different choices of the unstable subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ and $\mathbb{Z}_u^2 = \mathbb{Z}_u^1 \oplus G(\mu_6)$, and for three different values of the perturbation amplitude β. The rigidity coefficient is fixed at $\boldsymbol{\alpha = 10^{-2}}$, Reynolds number $Re = 200$ and damping coefficient $\gamma = 10^{-6}$.



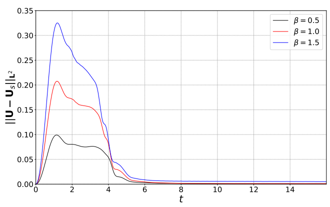
(a)



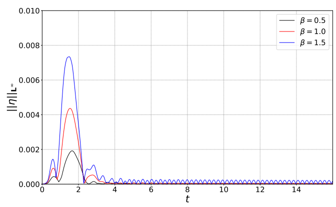
(b)



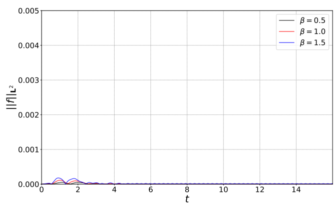
(c)



(d)



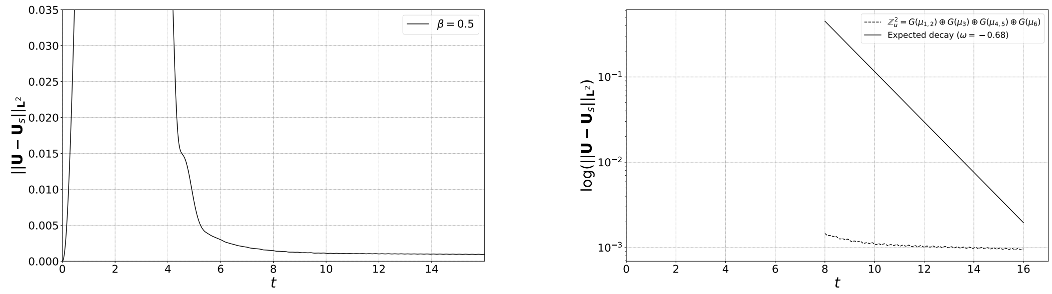
(e)



(f)

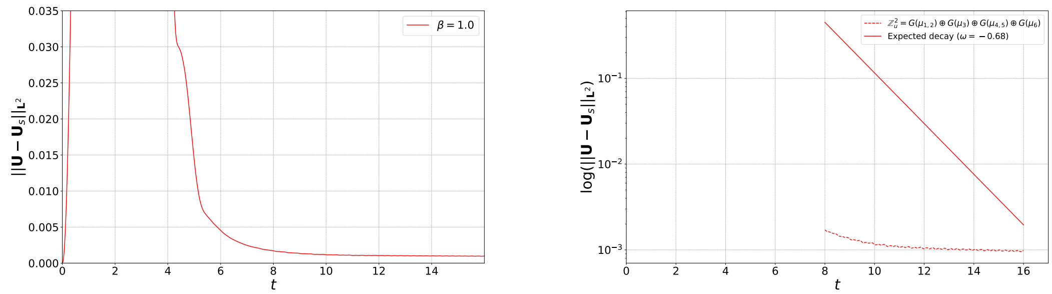
Figure 6.26 – Evolution of (a): $\|\mathbf{U}-\mathbf{U}_s\|_{\mathbf{L}^2}$, (b): $\|\eta\|_{L^\infty}$ and (c): $\|f\|_{L^2}$, for three different values of the amplitude perturbation β, when the rigidity coefficient $\alpha = 10^{-2}$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$. Row (a)-(b)-(c): Control based on $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$. Row (d)-(e)-(f): Control based on $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$.

As previously pointed out, in the case $\mathbb{Z}_u^2$ ($\dim(\mathbb{Z}_u^2) = 6$) and $\beta = 1.5$ (worst-case scenario), it appears that neither the fluid-velocity is stabilized around the stationary solution nor the structure's displacement. However, from Figure 6.26d and 6.26e (see black and red curves), it is not clear whether similar conclusions apply for $\beta = 0.5$ and $\beta = 1$. From the plots shown in Figures 6.27b and 6.28b, we observe a similar behavior compared to that presented in Figure 6.20b (see blue curve). By doing a zoom-in view on Figure 6.26e (see black and red curves), we observe from Figure 6.29 a similar type of oscillation in the displacement of the structure as in the case of $\beta = 1.5$. This indicates that the displacement of the structure is not stabilized around $\eta = 0$ when the feedback operator is based on $\mathbb{Z}_u^2$. Thus, in summary, in the case where the rigidity coefficient is $\alpha = 10^{-2}$ and the feedback control is based on $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$, neither the fluid velocity nor the stucture's displacement is stabilized. Furthermore, in contrast to the conclusions obtained for the cases where the rigidity coefficient α is 1 and 10^{-1}, the inclusion of the eigenspace $G(\mu_6)$ into the subspace $\mathbb{Z}_u^1 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5})$ to construct the feedbaw law does not improve the stabilization process (see also Table 6.6). On the contrary, it seems to deteriorate the bahavior. A possible explanation of this phenomenon is that the inclusion of the eigenspace $G(\mu_6)$ into the subspace $\mathbb{Z}_u^1$ may activate or amplify the nonlinear effects of the system.



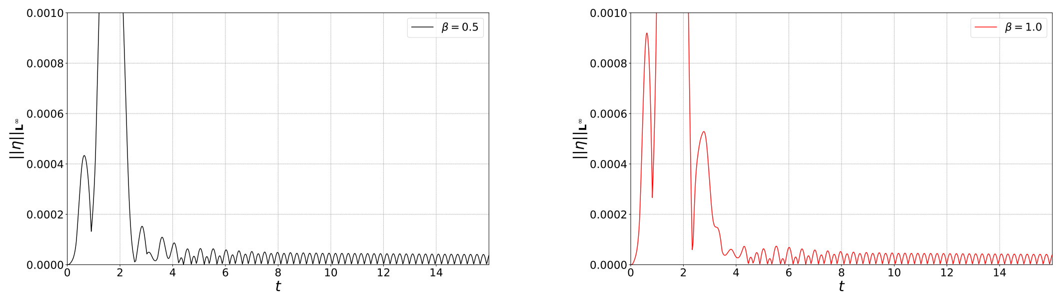
(a) (b)

Figure 6.27 – Behavior of $\|\mathbf{U} - \mathbf{U}_s\|_{L^2}$ in the case of rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 0.5$ when $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$. (a) Zoom-in view of the evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$. (b) Semi-log plot of $\|\mathbf{U} - \mathbf{U}_s\|_{L^2}$ versus t, illustrating exponential decay of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$.



(a) (b)

Figure 6.28 – Behavior of $\|\mathbf{U} - \mathbf{U}_s\|_{L^2}$ in the case of rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$, Reynolds number $Re = 200$ and perturbation amplitude $\beta = 1$ when $\mathbb{Z}_u^2 = G(\mu_{1,2}) \oplus G(\mu_3) \oplus G(\mu_{4,5}) \oplus G(\mu_6)$. (a) Zoom-in view of the evolution of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$. (b) Semi-log plot of $\|\mathbf{U} - \mathbf{U}_s\|_{L^2}$ versus t, illustrating exponential decay of $\|\mathbf{U} - \mathbf{U}_s\|_{\mathbf{L}^2}$.



(a) (b)

Figure 6.29 – Zoom-in view of the evolution of $\|\eta\|_{L^\infty}$ in the case of rigidity coefficient $\boldsymbol{\alpha = 10^{-2}}$, damping coefficient $\gamma = 10^{-6}$ and Reynolds number $Re = 200$. (a) Case perturbation amplitude $\beta = 0.5$. (b) Case perturbation amplitude $\beta = 1.0$.

Bibliographie

[BDDM07] Alain Bensoussan, Giuseppe Da Prato, Michel Delfour et Sanjoy Mitter. *Representation and Control of Infinite Dimensional Systems.* 2e éd. Birkhäuser Boston, 2007.

[Bei04] Hugo Beirão Da Veiga. « On the Existence of Strong Solutions to a Coupled Fluid-Structure Evolution Problem ». In : *J. Math. Fluid Mech.* 6 (2004), p. 21-52.

[BF12] Franck Boyer et Pierre Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models.* 1re éd. Springer New York, NY, 2012.

[BFGR] Franck Boyer, Michel Fournié, Diego Gajardo et Jean-Pierre. Raymond. « Study of the stationary Stokes system with mixed boundary conditions in non-convex polygonal domains ».

[BGM10] Leonardo Baffico, Céline Grandmont et Bertran Maury. « Multiscale Modeling of the respiratory tract ». In : *Mathematical Models and Methods in Applied Sciences* 20.01 (2010), p. 59-93.

[BH21] Ali Behzadan et Michael Holst. « Multiplication in Sobolev spaces, revisited ». In : *Arkiv för Matematik* 59 (2021), p. 275-306.

[BR] Mehdi Badra et Jean-Pierre Raymond. « The Oseen system with nonhomogeneous boundary conditions in polyhedral and polygonal domains ». https://hal.science/hal-04880963v2.

[BT17] Mehdi Badra et Takéo Takahashi. « Feedback boundary stabilization of 2d fluid-structure interaction systems ». In : *Discrete and Continuous Dynamical Systems* 37.5 (2017), p. 2315-2373.

[BT19] Mehdi Badra et Takéo Takahashi. « Gevrey Regularity for a System Coupling the Navier–Stokes System with a Beam Equation ». In : *SIAM Journal on Mathematical Analysis* 51.6 (2019), p. 4776-4814.

[CDEG05] Antonin Chambolle, Benoît Desjardins, Maria Esteban et Céline. Grandmont. « Existence of Weak Solutions for the Unsteady Interaction of a Viscous Fluid with an Elastic Plate ». In : *J. Math. Fluid Mech.* 7 (2005), p. 368-404.

[CGH21] Jean-Jérôme Casanova, Céline Grandmont et Matthieu Hillairet. « On an existence theory for a fluid-beam problem encompassing possible contacts ». en. In : *Journal de l'École polytechnique — Mathématiques* 8 (2021), p. 933-971.

[CR82] Goong Chen et David Lewis Russel. « A mathematical model for linear elastic systems with structural damping ». In : *Quart. Appl. Math.* 39 (1982), p. 433-454.

[CT89] Shu Ping Chen et Roberto Triggiani. « Proof of extensions of two conjectures on structural damping for elastic systems ». In : *Pacific J. Math.* 136(1) (1989), p. 15-55.

[Dau88] Monique Dauge. *Elliptic Boundary Value Problems on Corner Domains.* Springer-Verlag, Berlin, 1988.

[Dau89] Monique DAUGE. « Stationary Stokes and Navier–Stokes Systems on Two- or Three-Dimensional Domains with Corners. Part I. Linearized Equations ». In : *SIAM Journal on Mathematical Analysis* 20.1 (1989), p. 74-97.

[Del18] Guillaume DELAY. « Etude d'un problème d'interaction fluide-structure : modélisation, analyse, stabilisation et simulations numériques ». Theses. Institut de Mathématiques de Toulouse, 2018.

[Del19] Guillaume DELAY. « Local Stabilization of a Fluid–Structure System around a Stationary State with a Structure Given by a Finite Number of Parameters ». In : *SIAM Journal on Control and Optimization* 57.6 (2019), p. 4063-4098.

[DGH82] Jéan DONEA, Stéphane GIULIANI et Jean-Pierre HALLEUX. « An arbitrary lagrangian-eulerian finite element method for transient dynamic fluid-structure interactions ». In : *Computer Methods in Applied Mechanics and Engineering* 33.1 (1982), p. 689-723. ISSN : 0045-7825.

[DL99] Robert DAUTRAY et Jacques-Louis LIONS. *Mathematical Analysis and Numerical Methods for Science and Technology : Volume 5 Evolution Problems I.* Springer Science & Business Media, 1999.

[DPV12] Eleonora DI NEZZA, Giampiero PALATUCCI et Enrico VALDINOCI. « Hitchhiker guide to the fractional Sobolev spaces ». In : *Bulletin des Sciences Mathématiques* 136.5 (2012), p. 521-573. ISSN : 0007-4497aa.

[EG21] Alexandre ERN et Jean-Luc GUERMOND. *Finite Elements II : Galerkin Approximation, Elliptic and Mixed PDEs.* 1re éd. Springer Cham, 2021.

[EN06] Klaus-Jochen ENGEL et Rainer NAGEL. *A Short Course on Operator Semigroups.* Springer New York, NY, 2006.

[FAC] Michel FOURNIÉ, Franco AUTERI et Simone CRUCIANI. « Control of the Channel Flow Past a Cylinder by a Piezo-Actuated Flexible Splitter Plate ».

[FGG07] Miguel Ángel FERNÁNDEZ, Jean-Frédéric GERBEAU et Céline GRANDMONT. « A projection semi-implicit scheme for the coupling of an elastic structure with an incompressible fluid ». In : *International Journal for Numerical Methods in Engineering* 69.4 (2007), p. 794-821.

[FNR] Michel FOURNIÉ, Moctar NDIAYE et Jean-Pierre RAYMOND. « Numerical stabilization of a fluid-structure interaction system ».

[FNR19] Michel FOURNIÉ, Moctar NDIAYE et Jean-Pierre RAYMOND. « Feedback Stabilization of a Two-Dimensional Fluid-Structure Interaction System with Mixed Boundary Conditions ». In : *SIAM Journal on Control and Optimization* 57.5 (2019), p. 3322-3359.

[Fur01] Andreĭ Vladimirovich FURSIKOV. « Stabilizability of a quasi-linear parabolic equation by means of a boundary control with feedback ». In : *Sbornik : Mathematics* 192.4 (2001), p. 593-639.

[GR86] Vivette GIRAULT et Pierre-Arnaud RAVIART. *Finite Element Methods for Navier-Stokes Equations.* 1re éd. T. 5. Springer Berlin, Heidelberg, 1986.

[Gra08] Céline GRANDMONT. « Existence of Weak Solutions for the Unsteady Interaction of a Viscous Fluid with an Elastic Plate ». In : *SIAM Journal on Mathematical Analysis* 40.2 (2008), p. 716-737.

[GS11] Jean-Luc GUERMOND et Abner SALGADO. « A note on the Stokes operator and its powers ». In : *J. Appl. Math. Comput.* 36 (2011), p. 241-250.

[GS91] Gerd GRUBB et Vsevolod SOLONNIKOV. « Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods ». In : *Mathematica Scandinavica* 69.2 (1991), p. 217-290.

[HC23] Sang Truong HA et Hyoung Gwon CHOI. « Semi-monolithic formulation based on a projection method for simulating fluid–structure interaction problems ». In : *Computers & Mathematics with Applications* 134 (2023), p. 207-224. ISSN : 0898-1221.

[HP00] Antoine HENROT et Michel PIERRE. *Variation et optimisation de formes*. Springer, 2000.

[KR09] Srinivasan KESAVAN et Jean-Pierre RAYMOND. « On a degenerate Riccati equation ». In : *Control and Cybernetics* 38.4B (2009), p. 1393-1410.

[Leq11] Julien LEQUEURRE. « Existence of Strong Solutions to a Fluid-Structure System ». In : *SIAM Journal on Mathematical Analysis* 43.1 (2011), p. 389-410.

[LM72] Jacques-Louis LIONS et Enrico MAGENES. *Non-Homogeneous Boundary Value Problems and Applications*. T. 1. Springer Berlin, Heidelberg, 1972.

[MM08] Marius MITREA et Sylvie MONNIAUX. « The regularity of the Stokes operator and the Fujita–Kato approach to the Navier–Stokes initial value problem in Lipschitz domains ». In : *Journal of Functional Analysis* 254.6 (2008), p. 1522-1574. ISSN : 0022-1236.

[MR10] Vladimir MAZ'YA et Jürgen ROSSMANN. *Elliptic Equations in Polyhedral Domains*. American Mathematical Society, 2010.

[MR17] Debayan MAITY et Jean-Pierre RAYMOND. « Feedback Stabilization of the Incompressible Navier–Stokes Equations Coupled with a Damped Elastic System in Two Dimensions ». In : *J. Math. Fluid Mech.* 19 (2017), p. 773-805.

[MRR20] Debayan MAITY, Jean-Pierre RAYMOND et Arnab ROY. « Maximal-in-Time Existence and Uniqueness of Strong Solution of a 3D Fluid-Structure Interaction Model ». In : *SIAM Journal on Mathematical Analysis* 52.6 (2020), p. 6338-6378.

[Mur19] Cornel Marius MUREA. « Stable Semi-Implicit Monolithic Scheme for Interaction Between Incompressible Neo-Hookean Structure and Navier-Stokes Fluid ». In : *Journal of Mathematical Study* 52.4 (2019), p. 448-469. ISSN : 2617-8702.

[Ndi16] Moctar NDIAYE. « Stabilisation and simulation of fluid-structure interaction models ». Theses. Institut de Mathématiques de Toulouse, 2016.

[NR15] Phuong Anh NGUYEN et Jean-Pierre RAYMOND. « Boundary Stabilization of the Navier-Stokes Equations in the Case of Mixed Boundary Conditions ». In : *SIAM Journal on Control and Optimization* 53.5 (2015), p. 3006-3039.

[Paz83] Amnon PAZY. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. 1re éd. Springer, New York, NY, 1983.

[QF04] Alfio QUARTERONI et Luca FORMAGGIA. « Mathematical Modelling and Numerical Simulation of the Cardiovascular System ». In : *Computational Models for the Human Body*. T. 12. Handbook of Numerical Analysis. Elsevier, 2004, p. 3-127.

[QTV00] Alfio QUARTERONI, M Massimiliano TUVERI et Alessandro VENEZIANI. « Computational vascular fluid dynamics : problems, models and methods ». In : *Comput Visual Sci* 2 (2000), p. 163-197.

[Ray10] Jean-Pierre RAYMOND. « Feedback Stabilization of a Fluid-Structure Model ». In : *SIAM Journal on Control and Optimization* 48.8 (2010), p. 5398-5443.

[Ric15] Thomas RICHTER. « A monolithic geometric multigrid solver for fluid-structure interactions in ALE formulation ». In : *International Journal for Numerical Methods in Engineering* 104.5 (2015), p. 372-390.

[RP] Yves RENARD et Konstantinos POULIOS. « GetFEM : Automated FE modeling of multiphysics problems based on a generic weak form language ».

[RW10] Thomas Richter et Thomas Wick. « Finite elements for fluid–structure interaction in ALE and fully Eulerian coordinates ». In : *Computer Methods in Applied Mechanics and Engineering* 199.41 (2010), p. 2633-2642. ISSN : 0045-7825.

[SM08] Soyibou Sy et Cornel Marius Murea. « A stable time advancing scheme for solving fluid–structure interaction problem at small structural displacements ». In : *Computer Methods in Applied Mechanics and Engineering* 198.2 (2008), p. 210-222. ISSN : 0045-7825.

[Tar07] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces.* 1re éd. Springer Berlin, Heidelberg, 2007.

[TH06] Stefan Turek et Jaroslav Hron. « Proposal for Numerical Benchmarking of Fluid-Structure Interaction between an Elastic Object and Laminar Incompressible Flow ». In : *Fluid-Structure Interaction*. Springer Berlin Heidelberg, 2006, p. 371-385.

[Tri78] Hans Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Publishing Company, 1978.

[Wic11] Thomas Wick. « Fluid-structure interactions using different mesh motion techniques ». In : *Computers & Structures* 89.13 (2011), p. 1456-1467. ISSN : 0045-7949.

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Titre : Analyse d'un modèle d'interaction fluide-structure : caractère bien posé, stabilisation et simulations numériques

Mots clés : Modèle fluide-structure, Équation de Navier-Stokes incompressible, Stabilisation, Approche Euler Lagrangienne Arbitraire, Simulations numériques

Résumé : Cette thèse porte sur l'étude d'un modèle d'interaction fluide-structure qui couple les équations de Navier-Stokes dans un domaine rectangulaire bidimensionnel avec des conditions aux limites mixtes, et une structure régie par une équation d'Euler-Bernoulli amortie. La structure, supposée encastrée à une extrémité et libre à l'autre, est immergée dans le domaine fluide.
Dans la première partie de la thèse, nous établissons l'existence locale en temps de solutions fortes. La démonstration de ce résultat repose principalement sur trois éléments : la réécriture du système dans un domaine de référence fixé ; une analyse minutieuse du système de Stokes stationnaire avec des conditions aux limites mixtes dans un domaine présentant des coins rentrants, dans le cadre des espaces de Sobolev hétérogènes ; et enfin, des estimations ad hoc pour les termes non linéaires, lesquelles, combinées à un argument de point fixe, nous permettent de conclure le résultat.
La deuxième partie consiste à montrer que le système fluide-structure est localement exponentiellement stabilisable autour d'une solution stationnaire instable, avec un taux de décroissance arbitrairement prescrit, au moyen d'un contrôle par retour d'état appliqué comme terme de forçage dans l'équation de la structure. Bien que la démonstration et les difficultés soient similaires à celles rencontrées lors de l'établissement de l'existence de solutions fortes, des défis supplémentaires apparaissent, propres à la stabilisation du système linéarisé. Par exemple, l'analyse des problèmes aux valeurs propres direct et adjoint avec des conditions non standard.
Dans la troisième partie de la thèse, nous abordons les simulations numériques à la fois du problème direct et du problème de stabilisation. Pour résoudre le problème direct, nous utilisons un algorithme monolithique semi-implicite disponible dans la littérature. En ce qui concerne le problème de stabilisation semi-discret (en espace), il est possible de raisonner de manière similaire au cas continu. La principale nouveauté réside dans le fait que, pour réécrire le système dans le domaine de référence fixe, au lieu d'utiliser une transformation géométrique, nous considérons un prolongement harmonique approprié pour définir l'application. Plusieurs expériences numériques sont présentées.

Title: Analysis of a fluid-structure interaction model: well-posedness, stabilization and numerical simulations

Key words: Fluid-Structure Model, Incompressible Navier-Stokes equation, Stabilization, Arbitrary Lagrangian Euler Approach, Numerical simulations

Abstract: This thesis deals with the study of a fluid-structure interaction model that couples the Navier-Stokes equations in a two-dimensional rectangular domain with mixed boundary conditions, and a structure governed by a damped Euler-Bernoulli equation. The structure, which is assumed to be clamped at one end and free at the other one, is immersed in the fluid domain.
In the first part of the thesis, we establish the local-in-time existence of strong solutions. The proof of this result relies primarily on three main ingredients: rewriting the system in a fixed reference domain; a careful analysis of the stationary Stokes system with mixed boundary conditions in a domain with reentrant corners in the framework of heterogeneous Sobolev spaces; and finally, ad-hoc estimates for the nonlinear terms, which, together with a fixed-point argument, enable us to conclude the result.
The second part consists in showing that the fluid-structure system is exponentially stabilizable–locally around an unstable stationary solution–with any prescribed decay rate, using a feedback control applied as a forcing term in the structure equation. Although the proof and difficulties are similar to those encountered in establishing the existence of strong solutions, additional challenges arise that are intrinsic to the stabilization of the linearized system. For instance, the analysis of both the direct and adjoint eigenvalue problems involves non-standard conditions.
In the third part of the thesis, we address numerical simulations of both the forward problem and the stabilization problem. To solve the forward problem, we use a semi-implicit monolithic algorithm available in the literature. Regarding the semi-discrete (in space) stabilization problem, it is possible to argue in a manner similar to the continuous case. The main novelty lies in the fact that, to rewrite the system in the fixed referenced domain, instead of using a geometric transformation, we consider an appropriate harmonic extension to define the mapping. Several numerical experiments are presented.